Left invariant complex structures on $U(2)$ and $SU(2) \times SU(2)$ revisited. *

L. Magnin
Institut Mathématique de Bourgogne †
magnin@u-bourgogne.fr

December 13, 2008

Abstract

We compute the torsion-free linear maps $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$, deduce a new determination of the integrable complex structures and their equivalence classes under the action of the automorphism group for $\mathfrak{u}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and prove that in both cases the set of complex structures is a differentiable manifold. $\mathfrak{u}(2) \oplus \mathfrak{u}(2)$, $\mathfrak{su}(2)^N$ and $\mathfrak{u}(2)^N$ are also considered. Extension of complex structures from $\mathfrak{u}(2)$ to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are studied, local holomorphic charts given, and attention is paid to what representations of $\mathfrak{u}(2)$ we can get from a substitute to the regular representation on a space of holomorphic functions for the complex structure.

1 Introduction.

The left invariant integrable complex structures on the group $U(2)$ of unitary 2×2 matrices, i.e. integrable complex structures on its Lie algebra $\mathfrak{u}(2)$, have been computed for the first time, up to equivalence, in [11] in the algebraic approach, that is by determining the complex Lie subalgebras $\mathfrak{m}$ of the complexification of $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{u}(2)$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$, bar denoting conjugation. More recently, and more generally, all left invariant maximal rank CR-structures on any finite dimensional compact Lie group have been classified up to equivalence in [2]. Independently, in the case of $SU(2) \times SU(2)$, the integrable complex structures on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have been computed in [4] by direct approach and computations.

In the present paper, we first compute the torsion-free linear maps $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$. They appear to be maximal rank CR-structures, of the $CR0$-type in the classification of [2]. Then we show how to deduce, with the computer assisted methods of [6], a new determination of the integrable complex structures and their equivalence classes under the action of the automorphism group for the specific cases of $\mathfrak{u}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, without resorting to the general results of [2]. Our method consists in growing dimensions starting with torsion-free linear maps of $\mathfrak{su}(2)$.

* *Math. Subj. Class. [2000]* : 17B30

† UMR CNRS 5584, Université de Bourgogne, BP 47870, 21078 Dijon Cedex, France.

$\mathfrak{u}(2)\oplus\mathfrak{u}(2)$, and extensions for $\mathfrak{su}(2)^N$ and $\mathfrak{u}(2)^N$ are done too. In thoses cases, the set of complex structures is a differentiable manifold, though we write down explicit proofs only in the cases of $\mathfrak{u}(2)$ and $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. We also examine the extension of complex structures from $\mathfrak{u}(2)$ to $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, compute local complex charts for the complex manifolds associated to the complex structures, and determine what representations of $\mathfrak{u}(2)$ we can get from a substitute to the regular representation on a space of holomorphic functions for the complex structure.

2 Preliminaries.

Let G_0 be a connected finite dimensional real Lie group, with Lie algebra $\mathfrak{g}$. An almost complex structure on $\mathfrak{g}$ is a linear map $J:\mathfrak{g}\to\mathfrak{g}$ such that $J^2=-1$. The almost complex structure J is said to be *integrable* if it satisfies the condition

$$[JX,JY]-[X,Y]-J[JX,Y]-J[X,JY]=0\quad\forall X,Y\in\mathfrak{g}. \tag{1}$$

From the Newlander-Nirenberg theorem [10], condition (1) means that G_0 can be given the structure of a complex manifold with the same underlying real structure and such that the canonical complex structure on G_0 is the left invariant almost complex structure $\hat{J}$ associated to J. (For more details, see [6]). By a complex structure on $\mathfrak{g}$, we will mean an *integrable* almost complex structure on $\mathfrak{g}$, that is one satisfying (1).

Let J a complex structure on $\mathfrak{g}$ and denote by $G=(G_0,J)$ the group G_0 endowed with the structure of complex manifold defined by $\hat{J}$. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ splits as $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}^{(1,0)}\oplus\mathfrak{g}^{(0,1)}$ where $\mathfrak{g}^{(1,0)}=\{\tilde{X}=X-iJX;\ X\in\mathfrak{g}\}$, $\mathfrak{g}^{(0,1)}=\{\tilde{X}^-=X+iJX;\ X\in\mathfrak{g}\}$. We will denote $\mathfrak{g}^{(1,0)}$ by $\mathfrak{m}$. The integrability of J amounts to $\mathfrak{m}$ being a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In that way the set of complex structures on $\mathfrak{g}$ can be identified with the set of all complex subalgebras $\mathfrak{m}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}}=\mathfrak{m}\oplus\bar{\mathfrak{m}}$, bar denoting conjugation in $\mathfrak{g}_{\mathbb{C}}$. In particular, J is said to be abelian if $\mathfrak{m}$ is. That is the algebraic approach. Our approach is more trivial. We fix a basis of $\mathfrak{g}$, write down the torsion equations $ij|k$ $(1\leqslant i,j,k\leqslant n\ )$ obtained by projecting on x_k the equation $[Jx_i,Jx_j]-[x_i,x_j]-J[Jx_i,x_j]-J[x_i,Jx_j]=0$, where $(x_j)_{1\leqslant j\leqslant n}$ is the basis of $\mathfrak{g}$ we use, and solve them in steps by specific programs with the computer algebra system *Reduce* by A. Hearn. These programs, are downloadable in [3]. From now on, we will use the same notation J for J and $\hat{J}$ as well. For any $x\in G_0$, the complexification $T_x(G_0)_{\mathbb{C}}$ of the tangent space also splits as the direct sum of the holomorphic vectors $T_x(G_0)^{(1,0)}=\{\tilde{X}=X-iJX;\ X\in T_x(G_0)\}$ and the antiholomorphic vectors $T_x(G_0)^{(0,1)}=\{\tilde{X}^-=X+iJX;\ X\in T_x(G_0)\}$. For any open subset $V\subset G_0$, the space $H_{\mathbb{C}}(V)$ of complex valued holomorphic functions on V is comprised of all complex smooth functions f on V which are annihilated by any antiholomorphic vector field. This is equivalent to f being annihilated by all

$$\tilde{X}_j^-=X_j+iJX_j\quad 1\leqslant j\leqslant n \tag{2}$$

with $(X_j)_{1\leqslant j\leqslant n}$ the left invariant vector fields associated to the basis $(x_j)_{1\leqslant j\leqslant n}$ of $\mathfrak{g}$. Hence :

$$H_{\mathbb{C}}(V)=\{f\in C^\infty(V)\ ;\ \tilde{X}_j^-f=0\ \ \forall j\ \ 1\leqslant j\leqslant n\}. \tag{3}$$

Finally, the automorphism group Aut $\mathfrak{g}$ of $\mathfrak{g}$ acts on the set $\mathfrak{X}_{\mathfrak{g}}$ of all complex structures on $\mathfrak{g}$ by $J \mapsto \Phi \circ J \circ \Phi^{-1} \quad \forall \Phi \in \text{Aut } \mathfrak{g}$. Two complex structures J, J' on $\mathfrak{g}$ are said to be *equivalent* if they are on the same Aut $\mathfrak{g}$ orbit. For simply connected G_0, this amounts to requiring the existence of an $f \in \text{Aut } G_0$ such that $f : (G_0, J) \to (G_0, J')$ is biholomorphic.

3 $U(2)$.

Consider the Lie algebra $\mathfrak{su}(2)$ along with its basis $\{J_1, J_2, J_3\}$ defined by $J_1 = \frac{i}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $J_2 = \frac{1}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $J_3 = \frac{i}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. One has

$$[J_1, J_2] = J_3,\ [J_2, J_3] = J_1,\ [J_3, J_1] = J_2 \tag{4}$$

and the corresponding one-parameter subgroups of $SU(2)$ are $e^{tJ_1} = \left(\begin{smallmatrix} \cos\frac{t}{2} & i\sin\frac{t}{2} \\ i\sin\frac{t}{2} & \cos\frac{t}{2} \end{smallmatrix}\right)$, $e^{tJ_2} = \left(\begin{smallmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{smallmatrix}\right)$, $e^{tJ_3} = \left(\begin{smallmatrix} e^{i\frac{t}{2}} & 0 \\ 0 & e^{-i\frac{t}{2}} \end{smallmatrix}\right)$. By means of the basis $\{J_1, J_2, J_3\}$, $\mathfrak{su}(2)$ can be identified to the euclidean vector space $\mathbb{R}^3$ the bracket being then identified to the vector product $\wedge$. Then $Aut\,\mathfrak{su}(2)$ is comprised of the matrices $A = Mat(\mathbf{a}, \mathbf{b}, \mathbf{a} \wedge \mathbf{b})$ with $\mathbf{a}, \mathbf{b}$ any two orthogonal normed vectors in $\mathbb{R}^3$, i.e. $Aut\,\mathfrak{su}(2) \cong SO(3)$. Now, $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{c}$ where $\mathfrak{c} = \mathbb{R}\, J_4$ is the center of $\mathfrak{u}(2)$, $J_4 = \frac{i}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We use the basis (J_1, J_2, J_3, J_4) for $\mathfrak{u}(2)$. $\mathbb{R}^*$ stands for $\mathbb{R} \setminus \{0\}$.

Lemma 1. *$Aut\,\mathfrak{u}(2) \cong SO(3) \times \mathbb{R}^*$.*

Proof. As the center is invariant, any $\Phi \in Aut\,\mathfrak{u}(2)$ is of the form

$$\Phi = \begin{pmatrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_4^4 \end{pmatrix}$$

with $A \in Aut\,\mathfrak{su}(2) \cong SO(3)$ and $b_4^4 \in \mathbb{R}^*$. Necessarily, $b_1^4 = b_2^4 = b_3^4 = 0$, since $\Phi(J_k) \in [\mathfrak{u}(2), \mathfrak{u}(2)] = \mathfrak{su}(2)$ $(1 \leqslant k \leqslant 3)$. □

Lemma 2. *Let $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$ linear. J has zero torsion, i.e. satisfies (1), if and only if there exists $R \in SO(3)$ such that*

$$R^{-1} J R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi_3^3 \end{pmatrix}. \tag{5}$$

Proof. Let $J = (\xi^i_j)_{1\leqslant i,j\leqslant 3}$ in the basis (J_1, J_2, J_3). The 9 torsion equations are

$$\begin{array}{ll}
12|1 & \xi^1_3(\xi^2_2+\xi^1_1)+\xi^3_1(\xi^2_2-\xi^1_1)-\xi^3_2(\xi^2_1+\xi^1_2)=0,\\
12|2 & \xi^2_3(\xi^2_2+\xi^1_1)-\xi^3_2(\xi^2_2-\xi^1_1)-\xi^3_1(\xi^2_1+\xi^1_2)=0,\\
12|3 & \xi^1_2\xi^2_1-\xi^2_2\xi^1_1-(\xi^3_1)^2-(\xi^3_2)^2+\xi^3_3(\xi^2_2+\xi^1_1)+1=0,\\
13|1 & \xi^1_1(\xi^2_1-\xi^1_2)+\xi^2_3(\xi^1_3+\xi^3_1)-\xi^3_3(\xi^2_1+\xi^1_2)=0,\\
13|2 & \xi^1_3\xi^3_1+\xi^2_2\xi^1_1-(\xi^2_1)^2-(\xi^2_3)^2+\xi^3_3(\xi^2_2-\xi^1_1)+1=0,\\
13|3 & -\xi^1_1(\xi^2_3+\xi^3_2)+\xi^2_1(\xi^1_3+\xi^3_1)+\xi^3_3(\xi^2_3-\xi^3_2)=0,\\
23|1 & \xi^3_2\xi^2_3+\xi^2_2\xi^1_1-(\xi^1_3)^2-(\xi^1_2)^2-\xi^3_3(\xi^2_2-\xi^1_1)+1=0,\\
23|2 & \xi^2_2(\xi^2_1-\xi^1_2)-\xi^1_3(\xi^2_3+\xi^3_2)+\xi^3_3(\xi^2_1+\xi^1_2)=0,\\
23|3 & \xi^2_2(\xi^3_1+\xi^1_3)-\xi^1_2(\xi^2_3+\xi^3_2)+\xi^3_3(\xi^3_1-\xi^1_3)=0.
\end{array}$$

Again, we identify $\mathfrak{su}(2)$ to $\mathbb{R}^3$ with the vector product by means of the basis (J_1, J_2, J_3). J has at least one real eigenvalue λ. Let $\mathbf{f_3} \in \mathbb{R}^3$ some normed eigenvector associated to λ. Then there exist normed vectors $\mathbf{f_1}, \mathbf{f_2} \in \mathbb{R}^3$ such that $(\mathbf{f_1}, \mathbf{f_2}, \mathbf{f_3})$ is a direct orthonormal basis of $\mathbb{R}^3$. Hence there exists $R \in SO(3)$ such that

$$R^{-1}AR = \begin{pmatrix} * & * & 0\\ * & * & 0\\ * & * & \lambda\end{pmatrix}.$$

Hence we may suppose $\xi^1_3 = \xi^2_3$ in J. Now, the torsion equations $12|1$ and $12|2$ read respectively $\xi^3_1(\xi^2_2-\xi^1_1) = \xi^3_2(\xi^2_1+\xi^1_2)$, $\xi^3_1(\xi^2_1+\xi^1_2) = \xi^3_2(\xi^1_1-\xi^2_2)$ and imply the 2 equations $(\xi^3_1)^2(\xi^2_2-\xi^1_1) = -(\xi^3_2)^2(\xi^2_2-\xi^1_1)$, $(\xi^3_2)^2(\xi^2_1+\xi^1_2) = -(\xi^3_1)^2(\xi^2_1+\xi^1_2)$. Hence each one of the conditions $\xi^2_2 \neq \xi^1_1$ or $\xi^2_1 \neq -\xi^1_2$ implies $\xi^3_1 = \xi^3_2 = 0$. We now have 2 cases. Case 1 : $\xi^3_1 = \xi^3_2 = 0$. Case 2 : ξ^3_1, ξ^3_2 not both zero. In Case 2, one necessarily has $\xi^2_2 = \xi^1_1$ and $\xi^2_1 = -\xi^1_2$. Then equations $23|1$ and $23|2$ read $-(\xi^1_2)^2 + (\xi^1_1)^2 + 1 = 0$, $\xi^1_2\xi^1_1 = 0$ and give $\xi^1_1 = 0$, $\xi^1_2 = \pm 1$. Now equation $12|3$ reads $(\xi^3_2)^2 + (\xi^3_1)^2 = 0$. Hence Case 2 doesn't occur, i.e. one may suppose $\xi^3_1 = \xi^3_2 = 0$. Then equations $13|1$ and $23|2$ read resp. $\xi^3_3(\xi^2_1+\xi^1_2) = \xi^1_1(\xi^2_1-\xi^1_2)$, $\xi^3_3(\xi^2_1+\xi^1_2) = -\xi^2_2(\xi^2_1-\xi^1_2)$, hence if $\xi^2_1 \neq \xi^1_2$, necessarily $\xi^2_2 = -\xi^1_1$. Now $\xi^2_1 = \xi^1_2$ is impossible since it would imply either $\xi^3_3 = 0$ or $\xi^1_2 = 0$. In fact, first, if $\xi^1_2 = 0$, equations $12|3$, $13|2$ and $23|1$ read resp. $\xi^3_3(\xi^2_2+\xi^1_1) - \xi^2_2\xi^1_1 + 1 = 0$, $\xi^3_3(-\xi^2_2+\xi^1_1) - \xi^2_2\xi^1_1 - 1 = 0$, $\xi^3_3(-\xi^2_2+\xi^1_1) + \xi^2_2\xi^1_1 + 1 = 0$, so that $12|3 + 13|2$ gives $\xi^1_1(\xi^3_3 - \xi^2_2) = 0$ and $12|3 + 23|1$ gives $\xi^1_1\xi^3_3 = -1$, hence $\xi^1_1 \neq 0$ and $\xi^3_3 = \xi^2_2$, which is impossible since then $12|3$ reads $(\xi^2_2)^2 + 1 = 0$. Second, if $\xi^3_3 = 0$, $12|3$, $13|2$ read resp. $-\xi^2_2\xi^1_1 + (\xi^1_2)^2 + 1 = 0$, $-\xi^2_2\xi^1_1 + (\xi^1_2)^2 - 1 = 0$, which is contradictory. Hence we get as asserted $\xi^2_1 \neq \xi^1_2$ and $\xi^2_2 = -\xi^1_1$. Now we prove that $\xi^1_1 = 0$ and $\xi^1_2 = \pm 1$. Since $\xi^2_2 = -\xi^1_1$, equations $12|3$, $13|1$, $13|2$, $23|1$ read resp.

$$\begin{array}{ll}
12|3 & \xi^1_2\xi^2_1+(\xi^1_1)^2+1=0,\\
13|1 & \xi^3_3(\xi^2_1+\xi^1_2)-\xi^1_1(\xi^2_1-\xi^1_2)=0,\\
13|2 & 2\xi^3_3\xi^1_1+(\xi^2_1)^2+(\xi^1_1)^2-1=0,\\
23|1 & 2\xi^3_3\xi^1_1-(\xi^1_2)^2-(\xi^1_1)^2+1=0.
\end{array}$$

From $12|3$, $\xi_2^1 \neq 0$ and $\xi_1^2 = -\frac{1+(\xi_1^1)^2}{\xi_2^1}$. Then $13|1$, $13|2$, read resp. $Q = 0$, $R = 0$ with $Q = \xi_1^1((\xi_1^1)^2 + (\xi_2^1)^2 + 1) - \xi_3^3((\xi_1^1)^2 - (\xi_2^1)^2 + 1)$, $R = (\xi_2^1)^2(2\xi_3^3\xi_1^1 + (\xi_1^1)^2 - 1) + ((\xi_1^1)^2 + 1)^2$. Denote from $23|1$ $S = 2\xi_3^3\xi_1^1 - (\xi_2^1)^2 - (\xi_1^1)^2 + 1$. Suppose $\xi_1^1 \neq 0$. Then $N = \frac{R-S}{\xi_1^1} = 2\xi_3^3((\xi_2^1)^2 - 1) + \xi_1^1((\xi_1^1)^2 + (\xi_2^1)^2 + 3) = 0$ would give, for $\xi_2^1 \neq \pm 1$, $\xi_3^3 = -\frac{\xi_1^1((\xi_1^1)^2+(\xi_2^1)^2+3)}{2((\xi_2^1)^2-1)}$ and then $R = -\frac{((\xi_2^1)^2-2\xi_2^1+(\xi_1^1)^2+1)((\xi_2^1)^2+2\xi_2^1+(\xi_1^1)^2+1)}{(\xi_2^1)^2-1}$ which is impossible since the polynomial $X^2 \pm 2X + (\xi_1^1)^2 + 1$ has no real root. Hence $\xi_2^1 = \pm 1$. Now $S = 0$ gives $\xi_3^3 = \frac{\xi_1^1}{2}$ and then $R = (\xi_1^1)^2((\xi_1^1)^2 + 4) \neq 0$. Hence $\xi_1^1 = 0$. Finally that implies as asserted $\xi_2^1 = \pm 1$, since $R = -(\xi_2^1)^2 + 1$. We conclude that $\xi_1^1 = \xi_2^2 = 0, \xi_1^2 = -\xi_2^1, \xi_2^1 = \pm 1$. Changing if necessary Φ to $\Phi\,\mathrm{diag}(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), -1))$, one may suppose $\xi_2^1 = 1$. $\square$

Remark 1. Recall that a rank r CR-structure on a real Lie algebra $\mathfrak{g}$ is a r-dimensional subalgebra $\mathfrak{m}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ such that $\mathfrak{m} \cap \bar{\mathfrak{m}} = \{0\}$. Then $\mathfrak{m} = \{X - iJ_{\mathfrak{p}}X; X \in \mathfrak{p}\}$ where $\mathfrak{p}$ (the real part of $\mathfrak{m}$) is a vector subspace of $\mathfrak{g}$ and $J_{\mathfrak{p}} : \mathfrak{p} \to \mathfrak{p}$ is a zero torsion linear map such that $J_{\mathfrak{p}}^2 = -Id_{\mathfrak{p}}$. Alternatively, a CR-structure can be defined by such data $(\mathfrak{p}, J_{\mathfrak{p}})$. For even-dimensional $\mathfrak{g}$, CR-structures of maximal rank $r = \frac{1}{2}\dim\mathfrak{g}$ are just complex structures on $\mathfrak{g}$. CR-structures of maximal rank on a real compact Lie algebra have been classified in [2]. For odd-dimensional $\mathfrak{g}$, they fall essentially into 2 classes: $CR0$ and (strict) CRI. For even-dimensional $\mathfrak{g}$ they are all $CR0$. From lemma 2, any linear map $J : \mathfrak{su}(2) \to \mathfrak{su}(2)$ which has zero torsion is such that $\ker(J^2 + Id) \neq \{0\}$, and hence defines a maximal rank CR-structure on $\mathfrak{su}(2)$. It is of type $CR0$. Let us elaborate on that point. $\mathfrak{a}_0 = \mathbb{C}J_3$ is a Cartan subalgebra of $\mathfrak{su}(2)$. The complexification $\mathfrak{sl}(2)$ of $\mathfrak{su}(2)$ decomposes as $\mathfrak{sl}(2) = \mathbb{C}H_- \oplus \mathfrak{h} \oplus \mathbb{C}H_+$ with $H_\pm = iJ_1 \mp J_2$, $H_3 = iJ_3$, $\mathfrak{h} = \mathbb{C}H_3$. Any maximal rank CR-structure of $CR0$-type (resp. (strict) CRI-type) is equivalent to $\mathfrak{m} = \mathbb{C}H_+$ (resp. $\mathfrak{m} = \mathbb{C}(aJ_3 + H_+)$, $a \in \mathbb{R}^*$), and has real part $\mathfrak{p} = \mathbb{R}J_1 \oplus \mathbb{R}J_2$ (resp. $\mathfrak{p} = \mathbb{R}J_1 \oplus \mathbb{R}J_2'$, $J_2' = J_2 - aJ_3$). The corresponding endomorphism $J_{\mathfrak{p}}$ of $\mathfrak{p}$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the basis (J_1, J_2) (resp. (J_1, J_2')) and has zero torsion on $\mathfrak{p}$. Any extension of $J_{\mathfrak{p}}$ to $\mathfrak{su}(2)$ has matrix $\begin{pmatrix} 0 & 1 & \xi_3^1 \\ -1 & 0 & \xi_3^2 \\ 0 & 0 & \xi_3^3 \end{pmatrix}$ in the basis (J_1, J_2, J_3) (resp. (J_1, J_2', J_3)). In the $CR0$ case, it has zero torsion on the whole of $\mathfrak{su}(2)$ if and only if $\xi_3^1 = \xi_3^2 = 0$, i.e. is of the form (5). In the CRI case, it never has zero torsion on the whole of $\mathfrak{su}(2)$.

Lemma 3. *Let $\mathfrak{g} = \bigoplus_{j=1}^N \mathfrak{g}^{(j)}$, where $\mathfrak{g}^{(j)}$ are real Lie algebras with bases $\mathcal{B}_j = (X_k^{(j)})_{1 \leqslant k \leqslant n_j}$, and let $\pi^{(j)} : \mathfrak{g} \to \mathfrak{g}^{(j)}$ be the projections. Let $J : \mathfrak{g} \to \mathfrak{g}$ be a linear map, $\pi_j^i = \pi^{(i)} \circ J \circ \pi^{(j)}$, $\tilde{\pi}_j^i = \pi^{(i)} \circ J \circ \pi^{(j)}|_{\mathfrak{g}^{(j)}}$. If J has zero torsion, then the 2 following conditions are satisfied:*
(i) $\tilde{\pi}_i^i$ has zero torsion for any i;
(ii) $[\pi_j^i X, \pi_j^i Y] = \pi_j^i[JX, Y] + \pi_j^i[X, JY]$ $\forall X, Y \in \mathfrak{g}^{(j)}$ for any i, j such that $i \neq j$.

Proof. For any i, j let $X, Y \in \mathfrak{g}$. Applying $\pi^{(i)}$ to the torsion equation (1) we get

$$[\pi^{(i)}JX, \pi^{(i)}JY] - [\pi^{(i)}X, \pi^{(i)}Y] - \pi^{(i)}J[JX, Y] - \pi^{(i)}J[X, JY] = 0. \tag{6}$$

Suppose first $i = j$ and $X, Y \in \mathfrak{g}^{(i)}$. Then $[JX, Y] = [\pi^{(i)}JX, Y] = \pi^{(i)}[\pi^{(i)}J\pi^{(i)}X, Y]$, and $[X, JY] = [X, \pi^{(i)}JY] = \pi^{(i)}[X, \pi^{(i)}J\pi^{(i)}Y]$, and moreover $[\pi^{(i)}X, \pi^{(i)}Y] = [X, Y]$, hence

(6) gives

$$[\pi^{(i)}JX, \pi^{(i)}JY] - [X,Y] - \pi^{(i)}J\pi^{(i)}[\pi^{(i)}J\pi^{(i)}X, Y] - \pi^{(i)}J\pi^{(i)}[X, \pi^{(i)}J\pi^{(i)}Y],$$

i.e.

$$[\tilde{\pi}^i_i X, \tilde{\pi}^i_i Y] - [X,Y] - \tilde{\pi}^i_i[\tilde{\pi}^i_i X, Y] - \tilde{\pi}^i_i[X, \tilde{\pi}^i_i Y]$$

that is $\tilde{\pi}^i_i$ has no torsion. Suppose now $i \neq j$ and $X, Y \in \mathfrak{g}^{(j)}$. Then $[\pi^{(i)}X, \pi^{(i)}Y] = 0$ and (6) gives

$$[\pi^{(i)}JX, \pi^{(i)}JY] - \pi^{(i)}J\pi^{(j)}[JX, Y] - \pi^{(i)}J\pi^{(j)}[X, JY] = 0,$$

i.e.

$$[\pi^i_j X, \pi^i_j Y] - \pi^i_j[JX, Y] - \pi^i_j[X, JY] = 0.$$

□

Theorem 1. *(i) Let $J : \mathfrak{u}(2) \to \mathfrak{u}(2)$ linear. J has zero torsion, i.e. satisfies (1), if and only if there exists $\Phi \in SO(3) \times \mathbb{R}^*_+$ such that*

$$\Phi^{-1}J\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi^3_3 & \xi^3_4 \\ 0 & 0 & \xi^4_3 & \xi^4_4 \end{pmatrix}, \quad \begin{pmatrix} \xi^3_3 & \xi^3_4 \\ \xi^4_3 & \xi^4_4 \end{pmatrix} \in gl(2, \mathbb{R}). \tag{7}$$

(ii) Any $J \in \mathfrak{X}_{\mathfrak{u}(2)}$ is equivalent to a unique

$$J(\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi & 1 \\ 0 & 0 & -(1+\xi^2) & -\xi \end{pmatrix} \tag{8}$$

with $\xi \in \mathbb{R}$. $J(\xi)$ and $J(\xi')$ $(\xi, \xi' \in \mathbb{R})$ are equivalent if and only if $\xi = \xi'$.

Proof. (i) From lemma 3

$$J = \begin{pmatrix} & & & \xi^1_4 \\ & J_1 & & \xi^2_4 \\ & & & \xi^3_4 \\ \xi^4_1 & \xi^4_2 & \xi^4_3 & \xi^4_4 \end{pmatrix}$$

for some $J_1 : \mathfrak{su}(2) \to \mathfrak{su}(2)$ with zero torsion. From lemma 2, there exists $R \in SO(3)$ such that $R^{-1}J_1R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^3_3 \end{pmatrix}$ whence

$$\Phi^{-1}J\Phi = \begin{pmatrix} 0 & 1 & 0 & \xi^1_4 \\ -1 & 0 & 0 & \xi^2_4 \\ 0 & 0 & \xi^3_3 & \xi^3_4 \\ \xi^4_1 & \xi^4_2 & \xi^4_3 & \xi^4_4 \end{pmatrix}$$

$\Phi = \mathrm{diag}(R, 1)$. Hence we may suppose $J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^3_3 \end{pmatrix}$. Now the torsion equations $13|4, 23|4$ $14|3, 24|3$ give the 2 Cramer systems $\xi^4_2\xi^3_3 + \xi^4_1 = 0, -\xi^4_2 + \xi^3_3\xi^4_1 = 0; \xi^2_4\xi^3_3 - \xi^1_4 = 0, \xi^2_4 + \xi^3_3\xi^1_4 = 0.$

Hence $\xi_1^4 = \xi_2^4 = \xi_4^1 = \xi_4^2 = 0$. Then all torsion equations vanish, and (i) is proved ([3], `torsionu2.red`).
(ii) From (i), we may suppose

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & \xi_4^3 \\ 0 & 0 & \xi_3^4 & \xi_4^4 \end{pmatrix}.$$

Now $J \in \mathfrak{X}_{\mathfrak{u}(2)}$ if and only if $\begin{pmatrix} \xi_3^3 & \xi_4^3 \\ \xi_3^4 & \xi_4^4 \end{pmatrix}^2 = -I$, i.e.

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & \xi_4^3 \\ 0 & 0 & -\frac{1+(\xi_3^3)^2}{\xi_4^3} & -\xi_3^3 \end{pmatrix}, \quad \xi_4^3 \neq 0.$$

Observe now that for any $\Phi = \mathrm{diag}(A, b) \in Aut\, \mathfrak{u}(2) \quad (A \in SO(3), b \neq 0)$

$$\Phi J \Phi^{-1} = \begin{pmatrix} A J_1 A^{-1} & b^{-1} A \begin{pmatrix} 0 \\ 0 \\ \xi_4^3 \end{pmatrix} \\ b \begin{pmatrix} 0 & 0 & -\frac{1+(\xi_3^3)^2}{\xi_4^3} \end{pmatrix} A^{-1} & -\xi_3^3 \end{pmatrix}. \tag{9}$$

Taking $A = I$, $b = \xi_4^3$, we get

$$\Phi J \Phi^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & 1 \\ 0 & 0 & -(1+(\xi_3^3)^2) & -\xi_3^3 \end{pmatrix}.$$

Hence J is equivalent to $J(\xi)$ in (8) with $\xi = \xi_3^3$. The last assertion of the theorem results from (9). □

Remark 2. In [11], the equivalence classes of left invariant integrable complex structures on $\mathfrak{u}(2)$ are shown to be parametrized by the complex subalgebras $\mathfrak{m}_d$ with basis $\{J_1 + iJ_2, 2iJ_3 + dJ_4\}$ with $d = -\frac{1+i\xi}{1+\xi^2}$, $\xi \in \mathbb{R}$. The complex structure defined by $\mathfrak{m}_d$ has matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & \xi & 2(1+\xi^2) \\ 0 & 0 & -\frac{1}{2} & -\xi \end{pmatrix} = \Phi J(\xi) \Phi^{-1}$$

with $\Phi = \mathrm{diag}(1, 1, 1, \frac{1}{2(1+\xi^2)}) \in Aut\, \mathfrak{u}(2)$.

Remark 3. $\mathfrak{u}(2)$ has no abelian complex structures since, for $J(\xi)$, $\mathfrak{m} = \mathbb{C}\tilde{J}_1 \oplus \mathbb{C}\tilde{J}_3$ is the solvable Lie algebra $[\tilde{J}_1, \tilde{J}_3] = i(1 - i\xi)\tilde{J}_1$.

Corollary 1. *$\mathfrak{X}_{\mathrm{u}(2)}$ is comprised of the matrices*

$$\begin{pmatrix} (a_4^1)^2c^2\xi & (a_4^3+a_4^2a_4^1c\xi)c & (a_4^3a_4^1c\xi-a_4^2)c & a_4^1 \\ -(a_4^3-a_4^2a_4^1c\xi)c & (a_4^2)^2c^2\xi & (a_4^3a_4^2c\xi+a_4^1)c & a_4^2 \\ (a_4^3a_4^1c\xi+a_4^2)c & (a_4^3a_4^2c\xi-a_4^1)c & (a_4^3)^2c^2\xi & a_4^3 \\ -(\xi^2+1)c^2a_4^1 & -(\xi^2+1)c^2a_4^2 & -(\xi^2+1)c^2a_4^3 & -\xi \end{pmatrix} \tag{10}$$

with the conditions

$$\xi \in \mathbb{R}, \quad \begin{pmatrix} a_4^1 \\ a_4^2 \\ a_4^3 \end{pmatrix} \in \mathbb{R}^3 \setminus \{0\}, \quad c = \pm\left((a_4^1)^2+(a_4^2)^2+(a_4^3)^2\right)^{-\frac{1}{2}}. \tag{11}$$

Proof. As is known, any $R \in SO(3)$ can be written

$$\begin{pmatrix} u^2-v^2-w^2+s^2 & -2(uv+ws) & 2(-uw+sv) \\ 2(-sw+uv) & u^2-v^2+w^2-s^2 & -2(su+vw) \\ 2(sv+uw) & 2(su-vw) & u^2+v^2-w^2-s^2 \end{pmatrix} \tag{12}$$

for $q=(u,v,w,s) \in \mathbb{S}^3$ (R can be written in exactly 2 ways by means of q and $-q$). Hence any $\Phi \in Aut\,\mathfrak{u}(2)$ can be written

$$\Phi = \begin{pmatrix} & & & 0 \\ & R & & 0 \\ & & & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$$

with R as in (12) and $c \in \mathbb{R}^*$. Then we get for $\Phi J(\xi)\Phi^{-1}$ the matrix (10) with

$$a_4^1 = \frac{2}{c}(sv-uw), \tag{13}$$

$$a_4^2 = -\frac{2}{c}(su+vw), \tag{14}$$

$$a_4^3 = \frac{1}{c}(2u^2+2v^2-1). \tag{15}$$

From $u^2+v^2+w^2+s^2=1$, one gets $(a_4^1)^2+(a_4^2)^2+(a_4^3)^2=\frac{1}{c^2}$. Conversely, for any matrix J of the form (10) with conditions (11) there exist $\Phi \in Aut\,\mathfrak{u}(2)$ and $\xi \in \mathbb{R}$ such that $J=\Phi J(\xi)\Phi^{-1}$. This amounts to the existence of $q=(u,v,w,s)\in\mathbb{S}^3$ such that equations (13), (14), (15) hold true, and follows from the fact that the map $\mathbb{S}^3 \to \mathbb{S}^2$ $q \mapsto (ca_4^1, ca_4^2, ca_4^3)$ is the Hopf fibration. $\square$

Corollary 2. *$\mathfrak{X}_{\mathrm{u}(2)}$ is a closed 4-dimensional (smooth) submanifold of $\mathbb{R}^{16}$ with 2 connected components, each of them diffeomorphic to $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$.*

Proof. Denote $\mathfrak{X}^+_{\mathrm{u}(2)}$ (resp. $\mathfrak{X}^-_{\mathrm{u}(2)}$) the subset of those $J \in \mathfrak{X}_{\mathrm{u}(2)}$ with $c>0$ (resp. $c<0$). As c is uniquely defined by the matrix $J=(a_j^i) \in \mathfrak{X}_{\mathrm{u}(2)}$ by the formula

$$2c = \frac{a_4^3(a_2^1-a_1^2)+a_4^2(-a_3^1+a_1^3)+a_4^1(a_3^2-a_2^3)}{(a_4^1)^2+(a_4^2)^2+(a_4^3)^2},$$

one has $\mathfrak{X}_{\mathfrak{u}(2)} = \mathfrak{X}^+_{\mathfrak{u}(2)} \cup \mathfrak{X}^-_{\mathfrak{u}(2)}$ with disjoint union. $\mathfrak{X}^+_{\mathfrak{u}(2)}$ (resp. $\mathfrak{X}^-_{\mathfrak{u}(2)}$) is a closed subset of $\mathbb{R}^{16}$. It hence suffices to prove that $\mathfrak{X}^+_{\mathfrak{u}(2)}$ is a regular submanifold, the case of $\mathfrak{X}^-_{\mathfrak{u}(2)}$ being analogous. Let $F : \mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right) \to \mathfrak{X}^+_{\mathfrak{u}(2)}$ be the bijection defined by $F(\xi, (a^1_4, a^2_4, a^3_4)) = J$ where J is the matrix (10) with $c = \left((a^1_4)^2 + (a^2_4)^2 + (a^3_4)^2\right)^{-\frac{1}{2}}$. We equip $\mathfrak{X}^+_{\mathfrak{u}(2)}$ with the differentiable structure transferred from $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \{0\}\right)$. The injection i from $\mathfrak{X}^+_{\mathfrak{u}(2)}$ into the open subset $X \subset \mathbb{R}^{16}$ defined by $(a^1_4)^2 + (a^2_4)^2 + (a^3_4)^2 \neq 0$ is smooth. Now, there is a smooth retraction $r : X \mapsto \mathfrak{X}^+_{\mathfrak{u}(2)}$ defined by $r(A) = F(-a^4_4, (a^1_4, a^2_4, a^3_4))$ for $A = (a^i_j) \in X$. Hence i is an immersion and the topology of $\mathfrak{X}^+_{\mathfrak{u}(2)}$ is the induced topology of $\mathbb{R}^{16}$. $\square$

4 $SU(2) \times SU(2)$.

Lemma 4. *$Aut\,(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) = (SO(3) \times SO(3)) \cup \tau\,(SO(3) \times SO(3))$ where $\tau = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ is the switch between the two factors of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.*

Proof. Let $J^{(1)}_k$ $(1 \leqslant k \leqslant 3)$ (resp. $J^{(2)}_\ell$ $(1 \leqslant \ell \leqslant 3)$) be the basis for the first (resp. the second) factor $\mathfrak{su}(2)^{(1)}$ (resp. $\mathfrak{su}(2)^{(2)}$) of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with relations (4), and $\pi^{(1)}$ (resp. $\pi^{(2)}$) the corresponding projections. Let $\Phi = \left(\begin{smallmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{smallmatrix}\right) \in Aut\,(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$, each Φ_j being a 3×3 matrix. $\Phi_1 = \left(\pi^{(1)} \circ \Phi\right)_{|\mathfrak{su}(2)^{(1)}}$ is an homomorphism of $\mathfrak{su}(2)^{(1)}$ into itself. Hence the three columns of Φ_1 are two-by-two orthogonal vectors in $\mathbb{R}^3$ and if one of them is zero, then the 3 of them are zero. In particular, if $\Phi_1 \neq 0$, then $\Phi_1 \in SO(3)$. With the same reasoning, the same property holds true for Φ_2, Φ_3, Φ_4. Suppose first $\Phi_1 \neq 0$. For $k, \ell = 1, 2, 3$, $[\pi^{(1)}(\Phi(J^{(1)}_k)), \pi^{(1)}(\Phi(J^{(2)}_\ell))] = \pi^{(1)}(\Phi([J^{(1)}_k, J^{(2)}_\ell])) = 0$. That implies that any column of Φ_1 is colinear with any column of Φ_2, and hence $\Phi_2 = 0$ since the columns of Φ_1 are linearly independent. Then $\det \Phi_4 \neq 0$, whence $\Phi_4 \in SO(3)$ and finally $\Phi_3 = 0$ by the above reasoning. Hence $\Phi = \left(\begin{smallmatrix} \Phi_1 & 0 \\ 0 & \Phi_4 \end{smallmatrix}\right) \in SO(3) \times SO(3)$. Suppose now $\Phi_1 = 0$. Then $\det \Phi_2 \neq 0$, whence $\Phi_2 \in SO(3)$, and $\det \Phi_3 \neq 0$, whence $\Phi_3 \in SO(3)$. By the same argument as before, $\Phi_4 = 0$. Hence $\Phi = \left(\begin{smallmatrix} 0 & \Phi_2 \\ \Phi_3 & 0 \end{smallmatrix}\right) = \tau \left(\begin{smallmatrix} \Phi_3 & 0 \\ 0 & \Phi_2 \end{smallmatrix}\right) \in \tau\,(SO(3) \times SO(3))$. $\square$

Theorem 2. *Let $J : \mathfrak{su}(2) \oplus \mathfrak{su}(2) \to \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ linear. J has zero torsion, i.e. satisfies (1), if and only if there exists $\Phi \in SO(3) \times SO(3)$ such that*

$$\Phi^{-1} J \Phi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^3_3 & 0 & 0 & \xi^3_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \xi^6_3 & 0 & 0 & \xi^6_6 \end{pmatrix}. \tag{16}$$

Proof. From lemma 3 and lemma 2, there exists $\Phi \in SO(3) \times SO(3)$ such that

$$\Phi^{-1} J \Phi = \begin{pmatrix} 0 & 1 & 0 & \xi^1_4 & \xi^1_5 & \xi^1_6 \\ -1 & 0 & 0 & \xi^2_4 & \xi^2_5 & \xi^2_6 \\ 0 & 0 & \xi^3_3 & \xi^3_4 & \xi^3_5 & \xi^3_6 \\ \xi^4_1 & \xi^4_2 & \xi^4_3 & 0 & 1 & 0 \\ \xi^5_1 & \xi^5_2 & \xi^5_3 & -1 & 0 & 0 \\ \xi^6_1 & \xi^6_2 & \xi^6_3 & 0 & 0 & \xi^6_6 \end{pmatrix}. \tag{17}$$

Hence we may suppose J of the form (17). The matrix $(\xi^i_j)_{1\leqslant i\leqslant 3, 4\leqslant j\leqslant 6}$ (resp. $(\xi^i_j)_{4\leqslant i\leqslant 6, 1\leqslant j\leqslant 3}$) is the matrix of π^1_2 (resp. π^2_1) of lemma 3. Consider the vectors $\mathbf{u} = \pi^1_2 J^{(2)}_1$, $\mathbf{v} = \pi^1_2 J^{(2)}_2$, $\mathbf{w} = \pi^1_2 J^{(2)}_3$. From lemma 2 (ii) one has:

$$[\pi^1_2 J^{(2)}_1, \pi^1_2 J^{(2)}_2] = \pi^1_2 [JJ^{(2)}_1, J^{(2)}_2] + \pi^1_2 [J^{(2)}_1, JJ^{(2)}_2] = \pi^1_2 [-J^{(2)}_2, J^{(2)}_2] + \pi^1_2 [J^{(2)}_1, J^{(2)}_1] = 0,$$

$$\begin{aligned}[\pi^1_2 J^{(2)}_2, \pi^1_2 J^{(2)}_3] &= \pi^1_2 [JJ^{(2)}_2, J^{(2)}_3] + \pi^1_2 [J^{(2)}_2, JJ^{(2)}_3] = \pi^1_2 [J^{(2)}_1, J^{(2)}_3] + \pi^1_2 [J^{(2)}_2, \xi^6_6 J^{(2)}_3] \\ &= -\pi^1_2\, J^{(2)}_2 + \xi^6_6 \pi^1_2\, J^{(2)}_1,\end{aligned}$$

$$\begin{aligned}[\pi^1_2 J^{(2)}_1, \pi^1_2 J^{(2)}_3] &= \pi^1_2 [JJ^{(2)}_1, J^{(2)}_3] + \pi^1_2 [J^{(2)}_1, JJ^{(2)}_3] = \pi^1_2 [-J^{(2)}_2, J^{(2)}_3] + \pi^1_2 [J^{(2)}_1, \xi^6_6 J^{(2)}_3] \\ &= -\pi^1_2\, J^{(2)}_1 - \xi^6_6 \pi^1_2\, J^{(2)}_2.\end{aligned}$$

That is:

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} &= 0, \\ \mathbf{v} \wedge \mathbf{w} &= -\mathbf{v} + \xi^6_6 \mathbf{u}, \\ \mathbf{u} \wedge \mathbf{w} &= -\mathbf{u} - \xi^6_6 \mathbf{v}\end{aligned}$$

which implies $\mathbf{u} = \mathbf{v} = 0$. With the same reasoning for π^2_1, we get

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \xi^1_6 \\ -1 & 0 & 0 & 0 & 0 & \xi^2_6 \\ 0 & 0 & \xi^3_3 & 0 & 0 & \xi^3_6 \\ 0 & 0 & \xi^4_3 & 0 & 1 & 0 \\ 0 & 0 & \xi^5_3 & -1 & 0 & 0 \\ 0 & 0 & \xi^6_3 & 0 & 0 & \xi^6_6 \end{pmatrix}. \tag{18}$$

Now the torsion equations $16|3, 26|3\ 36|4, 36|5$ give the 2 Cramer systems $\xi^2_6\xi^3_3 - \xi^1_6 = 0$, $\xi^2_6 + \xi^3_3\xi^1_6 = 0$; $\xi^5_3\xi^6_6 + \xi^4_3 = 0$, $-\xi^5_3 + \xi^6_6\xi^4_3 = 0$. Hence $\xi^1_6 = \xi^2_6 = \xi^4_3 = \xi^5_3 = 0$. Then all torsion equations vanish, and the theorem is proved ([3], `torsionsu22.red`). □

Corollary 3. *Any $J \in \mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is equivalent under some member of $SO(3) \times SO(3)$ to*

$$J(\xi, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1+\xi^2}{\eta} & 0 & 0 & -\xi \end{pmatrix} \tag{19}$$

with $\xi, \eta \in \mathbb{R}$, $\eta \neq 0$. $J(\xi, \eta)$ and $J(\xi', \eta')$ are equivalent under some member of $SO(3) \times SO(3)$ (resp. $\tau\,(SO(3) \times SO(3))$) if and only if $\xi' = \xi$ and $\eta' = \eta$ (resp. $\xi' = -\xi$ and $\eta' = -\frac{1+\xi^2}{\eta}$).

Proof. J in (16) satisfies $J^2=-I$ if and only if $\xi_6^3\neq 0$ and $\xi_3^6=-\frac{1+(\xi_3^3)^2}{\xi_6^3}, \xi_6^6=-\xi_3^3$, leading to $J(\xi,\eta)$ in (19) with $\xi=\xi_3^3, \eta=\xi_6^3$.

Suppose $J(\xi',\eta')=\Phi J(\xi,\eta)\Phi^{-1}$ with $\Phi=\left(\begin{smallmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{smallmatrix}\right)\in SO(3)\times SO(3)$. Then $\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&\xi'\end{smallmatrix}\right)=\Phi_1\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&\xi\end{smallmatrix}\right)\Phi_1^{-1}$ and $\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&-\xi'\end{smallmatrix}\right)=\Phi_2\left(\begin{smallmatrix}0&1&0\\-1&0&0\\0&0&-\xi\end{smallmatrix}\right)\Phi_2^{-1}$, which imply first $\xi'=\xi$ and second $\Phi_1=\mathrm{diag}(R_1,1)$, $\Phi_2=\mathrm{diag}(R_2,1)$ with $R_1,R_2\in SO(2)$. Then $\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&\eta'\end{smallmatrix}\right)=\Phi_1\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&\eta\end{smallmatrix}\right)\Phi_2^{-1}$ implies $\eta'=\eta$.

Now suppose $J(\xi',\eta')=\Psi J(\xi,\eta)\Psi^{-1}$ with $\Psi=\tau\Phi\in\tau\left(SO(3)\times SO(3)\right)$, $\Phi=\left(\begin{smallmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{smallmatrix}\right)\in SO(3)\times SO(3)$. Then $\Phi J(\xi,\eta)\Phi^{-1}=\tau J(\xi',\eta')\tau=J(-\xi',-\frac{1+\xi'^2}{\eta'})$. Hence $\xi=-\xi'$ and $\eta=-\frac{1+\xi'^2}{\eta'}$, i.e. $\eta'=-\frac{1+\xi^2}{\eta}$. $\square$

Remark 4. Lemma 1 in [4] states that a left invariant almost complex structure on $SU(2)\times SU(2)$ is integrable if and only if it has the form $AI_{a,c}A^{-1}$ with $A\in SO(3)\times SO(3)$, $a\in\mathbb{R}, c\in\mathbb{R}^*$, and

$$I_{a,c}=\begin{pmatrix}\frac{a}{c} & 0 & 0 & -\frac{a^2+c^2}{c} & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \frac{1}{c} & 0 & 0 & -\frac{a}{c} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}.$$

One has $\Phi^{-1}I_{a,c}\Phi=J(\frac{a}{c},-\frac{a^2+c^2}{c})$ with $\Phi=\mathrm{diag}\left(\left(\begin{smallmatrix}0&0&-1\\0&1&0\\1&0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0&-1\\0&1&0\\1&0&0\end{smallmatrix}\right)\right)\in SO(3)\times SO(3)$.

Remark 5. $\mathfrak{su}(2)\times\mathfrak{su}(2)$ has no abelian complex structures since, for $J(\xi,\eta)$, $\mathfrak{m}=\mathbb{C}\tilde{J}_1^{(1)}\oplus\mathbb{C}\tilde{J}_3^{(1)}\oplus\mathbb{C}\tilde{J}_1^{(2)}$ is the solvable Lie algebra $[\tilde{J}_1^{(1)},\tilde{J}_3^{(1)}]=i(1-i\xi)\tilde{J}_1^{(1)}$, $[\tilde{J}_3^{(1)},\tilde{J}_1^{(2)}]=\frac{1+\xi^2}{\eta}\tilde{J}_1^{(2)}$.

Corollary 4. *$\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is comprised of the matrices*

$$\begin{pmatrix}\lambda_1^2\xi & -\lambda_1\mu_1\xi+\nu_1 & \lambda_1\nu_1\xi+\mu_1 & \eta\lambda_1\lambda_2 & -\eta\lambda_1\mu_2 & \eta\lambda_1\nu_2\\ -\lambda_1\mu_1\xi-\nu_1 & \mu_1^2\xi & \lambda_1-\mu_1\nu_1\xi & -\eta\mu_1\lambda_2 & \eta\mu_1\mu_2 & -\eta\mu_1\nu_2\\ \lambda_1\nu_1\xi-\mu_1 & -\lambda_1-\mu_1\nu_1\xi & \nu_1^2\xi & \eta\nu_1\lambda_2 & -\eta\nu_1\mu_2 & \eta\nu_1\nu_2\\ -\frac{\xi^2+1}{\eta}\lambda_1\lambda_2 & \frac{\xi^2+1}{\eta}\mu_1\lambda_2 & -\frac{\xi^2+1}{\eta}\nu_1\lambda_2 & -\lambda_2^2\xi & \lambda_2\mu_2\xi+\nu_2 & -\lambda_2\nu_2\xi+\mu_2\\ \frac{\xi^2+1}{\eta}\lambda_1\mu_2 & -\frac{\xi^2+1}{\eta}\mu_1\mu_2 & \frac{\xi^2+1}{\eta}\nu_1\mu_2 & \lambda_2\mu_2\xi-\nu_2 & -\mu_2^2\xi & \lambda_2+\mu_2\nu_2\xi\\ -\frac{\xi^2+1}{\eta}\lambda_1\nu_2 & \frac{\xi^2+1}{\eta}\mu_1\nu_2 & -\frac{\xi^2+1}{\eta}\nu_1\nu_2 & -\lambda_2\nu_2\xi-\mu_2 & -\lambda_2+\mu_2\nu_2\xi & -\nu_2^2\xi\end{pmatrix} \tag{20}$$

with

$$(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*,\quad \begin{pmatrix}\lambda_i\\ \mu_i\\ \nu_i\end{pmatrix}\in\mathbb{S}^2\quad i=1,2. \tag{21}$$

Proof. $\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is comprised of the matrices $\Phi J(\xi,\eta)\Phi^{-1}$, $(\xi,\eta)\in\mathbb{R}\times\mathbb{R}^*$, $\Phi\in SO(3)\times SO(3)$. Let $\Phi=\left(\begin{smallmatrix}\Phi_1 & 0\\ 0 & \Phi_2\end{smallmatrix}\right)\in SO(3)\times SO(3)$. Φ_1,Φ_2 can be written in the form (12) for resp. $q_1=(u_1,v_1,w_1,s_1)$, $q_2=(u_2,v_2,w_2,s_2)\in\mathbb{S}^3$. Then $\Phi J(\xi,\eta)\Phi^{-1}$ is the matrix (20) with for

$i = 1, 2$

$$\lambda_i = 2(s_i v_i - u_i w_i), \tag{22}$$
$$\mu_i = 2(s_i u_i + v_i w_i), \tag{23}$$
$$\nu_i = 2u_i^2 + 2v_i^2 - 1. \tag{24}$$

One has $\lambda_i^2 + \mu_i^2 + \nu_i^2 = 1$. Conversely, for any matrix J of the form (20) with condition (21) there exist $\Phi \in SO(3) \times SO(3)$ and $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^*$ such that $J = \Phi J(\xi, \eta)\Phi^{-1}$. This amounts to the existence for $i = 1, 2$ of $q_i = (u_i, v_i, w_i, s_i) \in \mathbb{S}^3$ such that equations (22), (23), (24) hold true, which again follows from the Hopf fibration. □

Corollary 5. *$\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is a closed 6-dimensional (smooth) submanifold of $\mathbb{R}^{36}$ diffeomorphic to $\mathbb{R} \times \mathbb{R}^* \times \left(\mathbb{S}^2\right)^2$.*

Proof. Let X the open subset of $\mathbb{R}^{36}$ of those matrices $\left(a_j^i\right)_{1\leqslant i,j\leqslant 6}$ such that $H^2 N_1 N_2 \neq 0$ where $H^2 = \sum_{i=1}^3 \sum_{j=4}^6 \left(a_j^i\right)^2$, $N_1 = \left(a_3^2 - a_2^3\right)^2 + \left(a_3^1 - a_1^3\right)^2 + \left(a_2^1 - a_1^2\right)^2$, $N_2 = \left(a_6^5 - a_5^6\right)^2 + \left(a_6^4 - a_4^6\right)^2 + \left(a_5^4 - a_4^5\right)^2$ and consider $F : \mathbb{R} \times \mathbb{R}^* \times \left(\mathbb{S}^2\right)^2 \to X$ defined by $F(\xi, \eta, (\lambda_1, \mu_1, \nu_1), (\lambda_2, \mu_2, \nu_2)) = J$ where J is the matrix (20).

Observe first that F is injective. In fact, $\xi, (\lambda_1, \mu_1, \nu_1), (\lambda_2, \mu_2, \nu_2)$ can be retrieved from $(a_j^i) = F(\xi, \eta, (\lambda_1, \mu_1, \nu_1), (\lambda_2, \mu_2, \nu_2))$ by the formulas $\xi = a_1^1 + a_2^2 + a_3^3$, $(\lambda_1, \mu_1, \nu_1) = \left(\frac{a_3^2 - a_2^3}{\sqrt{N_1}}, \frac{a_3^1 - a_1^3}{\sqrt{N_1}}, \frac{a_2^1 - a_1^2}{\sqrt{N_1}}\right)$, $(\lambda_2, \mu_2, \nu_2) = \left(\frac{a_6^5 - a_5^6}{\sqrt{N_2}}, \frac{a_6^4 - a_4^6}{\sqrt{N_2}}, \frac{a_5^4 - a_4^5}{\sqrt{N_2}}\right)$; hence $F(\xi, \eta, (\lambda_1, \mu_1, \nu_1), (\lambda_2, \mu_2, \nu_2)) = F(\xi', \eta', (\lambda_1', \mu_1', \nu_1'), (\lambda_2', \mu_2', \nu_2'))$ implies $\xi = \xi'$, $(\lambda_1', \mu_1', \nu_1') = (\lambda_1, \mu_1, \nu_1), (\lambda_2', \mu_2', \nu_2') = (\lambda_2, \mu_2, \nu_2)$, and then $\eta = \eta'$ since $\begin{pmatrix} \lambda_1\lambda_2 & -\lambda_1\mu_2 & \lambda_1\nu_2 \\ -\mu_1\lambda_2 & \mu_1\mu_2 & -\mu_1\nu_2 \\ \nu_1\lambda_2 & -\nu_1\mu_2 & \nu_1\nu_2 \end{pmatrix} \neq 0$. From the injectivity of F, $\mathfrak{X}_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} = \mathfrak{X}^+_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} \cup \mathfrak{X}^-_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ with disjoint union, where $\mathfrak{X}^\epsilon_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ denotes the set of those Js having η the sign of ϵ ($\epsilon = \pm$). Now, the map $G_\epsilon : X \to \mathbb{R} \times \mathbb{R}^*_\epsilon \times \left(\mathbb{S}^2\right)^2$ defined by

$$G_\epsilon(\left(a_j^i\right)) = \left(a_1^1 + a_2^2 + a_3^3, \epsilon\sqrt{\sum_{i=1}^{3}\sum_{j=4}^{6}\left(a_j^i\right)^2}, \left(\frac{a_3^2 - a_2^3}{\sqrt{N_1}}, \frac{a_3^1 - a_1^3}{\sqrt{N_1}}, \frac{a_2^1 - a_1^2}{\sqrt{N_1}}\right), \left(\frac{a_6^5 - a_5^6}{\sqrt{N_2}}, \frac{a_6^4 - a_4^6}{\sqrt{N_2}}, \frac{a_5^4 - a_4^5}{\sqrt{N_2}}\right)\right) \tag{25}$$

is a smooth retraction for the restriction F_ϵ of F to $\mathbb{R} \times \mathbb{R}^*_\epsilon \times \left(\mathbb{S}^2\right)^2$. Hence F_ϵ is an immersion and the topology of $\mathfrak{X}^\epsilon_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$ is the induced topology from X. The corollary follows. □

Remark 6. We may consider $\mathfrak{u}(2)$ as a subalgebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ by identifying J_1, J_2, J_3, J_4 to $J_1^{(1)}, J_2^{(1)}, J_3^{(1)}, J_3^{(2)}$ respectively. Then the complex structure J in (20) leaves $\mathfrak{u}(2)$ invariant if and only if $\lambda_2 = \mu_2 = 0, \nu_2 = \pm 1$. For the restriction of J to $\mathfrak{u}(2)$ to be (10), one must take $\lambda_1 = \frac{a_4^1}{\eta\nu_2}, \mu_1 = -\frac{a_4^2}{\eta\nu_2}, \nu_1 = \frac{a_4^3}{\eta\nu_2}$ with $c = \frac{\nu_2}{\eta}$. Then

$$J = \begin{pmatrix} (a_4^1)^2 c^2 \xi & (a_4^3 + a_4^2 a_4^1 c\xi)c & (a_4^3 a_4^1 c\xi - a_4^2)c & 0 & 0 & a_4^1 \\ -(a_4^3 - a_4^2 a_4^1 c\xi)c & (a_4^2)^2 c^2 \xi & (a_4^3 a_4^2 c\xi + a_4^1)c & 0 & 0 & a_4^2 \\ (a_4^3 a_4^1 c\xi + a_4^2)c & (a_4^3 a_4^2 c\xi - a_4^1)c & (a_4^3)^2 c^2 \xi & 0 & 0 & a_4^3 \\ 0 & 0 & 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & -\nu_2 & 0 & 0 \\ -(\xi^2+1)c^2 a_4^1 & -(\xi^2+1)c^2 a_4^2 & -(\xi^2+1)c^2 a_4^3 & 0 & 0 & -\xi \end{pmatrix}.$$

Hence any complex structure on $\mathfrak{u}(2)$ can be extended in 2 (in general non equivalent) ways to a complex structure on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. For example, $J(\xi)$ can be extended (here $a_4^1 = a_4^2 = 0, a_4^3 = 1, c = 1$) with $\nu_2 = 1$ to $J(\xi, 1)$ or with $\nu_2 = -1$ to $\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(1+\xi^2) & 0 & 0 & -\xi \end{smallmatrix}\right)$ which is equivalent to $J(\xi, -1)$. Now, $J(\xi, -1) \cong J(\xi, 1) \Leftrightarrow \xi = 0$.

5 $SU(2)^N$.

The results of lemma 4, theorem 2 and corollary 3 easily generalize in the following way.

Lemma 5. *For any $N \in \mathbb{N}^*$, $Aut\,(\mathfrak{su}(2))^N = SO(3)^N \cup \left(\bigcup_{\sigma\in\Sigma} \tau_\sigma \left(SO(3)^N\right)\right)$ (disjoint reunion) where Σ is the set of circular permutations of $\{1, \cdots, N\}$ having no fixed point, and $\tau_\sigma = (T_j^i)_{1\leqslant i,j\leqslant N}$ with the T_j^is the 3×3 blocks $T_j^i = \delta_{i,\sigma(j)} I$ (I the 3×3 identity and $\delta_{k,\ell}$ the Kronecker symbol).*

Theorem 3. *Let $J : \mathfrak{su}(2)^N \to \mathfrak{su}(2)^N$ linear. J has zero torsion if and only if there exists $\Phi \in SO(3)^N$ and $M = (\xi_{3j}^{3i})_{1\leqslant i,j\leqslant N} \in gl(N, \mathbb{R})$ such that $\Phi^{-1} J \Phi = J(M)$ with $J(M) = (J_j^i(M))_{1\leqslant i,j\leqslant N}$ and the $J_j^i(M)$s the following 3×3 blocks*

$$J_i^i(M) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi_{3i}^{3i} \end{pmatrix} \quad (1 \leqslant i \leqslant N)\,, \quad J_j^i(M) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_{3j}^{3i} \end{pmatrix} \quad (1 \leqslant i, j \leqslant N,\ i \neq j). \tag{26}$$

(Here we used that the analogs of 16|3, 26|3 36|4, 36|5 at the end of the proof of Theorem 2 are resp. , with $i < j$,

$$\begin{aligned} 3i-2, 3j|3i: &\quad \xi_{3j}^{3i-2} - \xi_{3i}^{3i}\xi_{3j}^{3i-1} = 0 \\ 3i-1, 3j|3i: &\quad \xi_{3j}^{3i-1} + \xi_{3i}^{3i}\xi_{3j}^{3i-2} = 0 \\ 3i, 3j|3j-2: &\quad -\xi_{3i}^{3j-2} - \xi_{3j}^{3j}\xi_{3i}^{3j-1} = 0 \\ 3i-1, 3j|3j-1: &\quad \xi_{3i}^{3j-1} + \xi_{3j}^{3j}\xi_{3i}^{3j-2} = 0 \end{aligned}$$

and give $\xi_{3j}^{3i-2} = \xi_{3j}^{3i-1} = \xi_{3i}^{3j-2} = \xi_{3i}^{3j-1} = 0$. Then all torsion equations vanish.)

Example 1. For $N=4$,

$$J(M)=\left(\begin{array}{ccc|ccc||ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \boxed{\xi_3^3} & 0 & 0 & \boxed{\xi_6^3} & 0 & 0 & \boxed{\xi_9^3} & 0 & 0 & \boxed{\xi_{12}^3}\\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \boxed{\xi_3^6} & 0 & 0 & \boxed{\xi_6^6} & 0 & 0 & \boxed{\xi_9^6} & 0 & 0 & \boxed{\xi_{12}^6}\\
\hline\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \boxed{\xi_3^9} & 0 & 0 & \boxed{\xi_6^9} & 0 & 0 & \boxed{\xi_9^9} & 0 & 0 & \boxed{\xi_{12}^9}\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & \boxed{\xi_3^{12}} & 0 & 0 & \boxed{\xi_6^{12}} & 0 & 0 & \boxed{\xi_9^{12}} & 0 & 0 & \boxed{\xi_{12}^{12}}
\end{array}\right)$$

$$M=\begin{pmatrix}
\xi_3^3 & \xi_6^3 & \xi_9^3 & \xi_{12}^3\\
\xi_3^6 & \xi_6^6 & \xi_9^6 & \xi_{12}^6\\
\xi_3^9 & \xi_6^9 & \xi_9^9 & \xi_{12}^9\\
\xi_3^{12} & \xi_6^{12} & \xi_9^{12} & \xi_{12}^{12}
\end{pmatrix}.$$

Corollary 6. *For even N, any $J\in\mathfrak{X}_{\mathfrak{su}(2)^N}$ is equivalent under some member of $SO(3)^N$ to some $J(M)=(J^i_j(M))_{1\leqslant i,j\leqslant N}$ with $M=\left(\xi^{3i}_{3j}\right)_{1\leqslant i,j\leqslant N}$ such that $M^2=-I$ and $J^i_j(M)$ defined in (26). $J(M)$ and $J(M')$ are equivalent under some member of $SO(3)^N$ (resp. $\tau_\sigma\left(SO(3)^N\right)$, $\sigma\in\Sigma$) if and only if $M'=M$ (resp. $M'=M^{\sigma^{-1}}$, $M^{\sigma^{-1}}=\left(\xi^{3\,\sigma^{-1}(i)}_{3\,\sigma^{-1}(j)}\right)_{1\leqslant i,j\leqslant N}$ (here we make use of $(\tau_\sigma)^{-1}=\tau_{\sigma^{-1}}$ and $\tau_\sigma J(M')(\tau_\sigma)^{-1}=\left(J^{\sigma^{-1}(i)}_{\sigma^{-1}(j)}(M')\right)_{1\leqslant i,j\leqslant N}$).*

Example 2. *For $N=2$, Σ consists only of the transposition $(1,2)$; $M=\begin{pmatrix}\xi_3^3 & \xi_6^3\\ -\frac{1+(\xi_3^3)^2}{\xi_6^3} & -\xi_3^3\end{pmatrix}$, $M'=\begin{pmatrix}{\xi'}_3^3 & {\xi'}_6^3\\ -\frac{1+({\xi'}_3^3)^2}{{\xi'}_6^3} & -{\xi'}_3^3\end{pmatrix}$. For $\sigma=(1,2)$, the condition $M'=M^{\sigma^{-1}}$ reads ${\xi'}_3^3=-\xi_3^3$, ${\xi'}_6^3=-\frac{1+(\xi_3^3)^2}{\xi_6^3}$ and is that of Corollary 3.*

6 $U(2)\times U(2)$.

Lemma 6. *$Aut\,(\mathfrak{u}(2)\oplus\mathfrak{u}(2))=H\cup\tau H$ where $\tau=\left(\begin{array}{c|c}0 & I\\ \hline I & 0\end{array}\right)$ is the switch between the two factors of $\mathfrak{u}(2)\oplus\mathfrak{u}(2)$, $H=\left\{\left(\begin{array}{cc|cc}\Phi_1 & 0 & 0 & 0\\ 0 & b_4^4 & 0 & b_8^4\\ \hline 0 & 0 & \Phi_4 & 0\\ 0 & b_4^8 & 0 & b_8^8\end{array}\right), \Phi_1,\Phi_4\in SO(3), b_4^4b_8^8-b_8^4b_4^8\neq 0\right\}$,*

$$
\tau H = \left\{ \left(\begin{array}{cc|cc} 0 & 0 & \Phi_2 & 0 \\ 0 & b_4^4 & 0 & b_8^4 \\ \hline \Phi_3 & 0 & 0 & 0 \\ 0 & b_4^8 & 0 & b_8^8 \end{array} \right), \Phi_2, \Phi_3 \in SO(3), b_4^4 b_8^8 - b_8^4 b_4^8 \neq 0 \right\}.
$$

Proof. Analog to that of lemma 4. □

Theorem 4. *(i) Let $J : \mathfrak{u}(2) \oplus \mathfrak{u}(2) \to \mathfrak{u}(2) \oplus \mathfrak{u}(2)$ linear. J has zero torsion if and only if there exists $\Phi \in \left(SO(3) \times \mathbb{R}_+^*\right)^2 \subset H$ and $M \in gl(4, \mathbb{R})$ such that $\Phi^{-1} J \Phi = K(M)$, where*

$$
K(M) = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & \xi_4^3 & 0 & 0 & \xi_7^3 & \xi_8^3 \\ 0 & 0 & \xi_3^4 & \xi_4^4 & 0 & 0 & \xi_7^4 & \xi_8^4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^7 & \xi_4^7 & 0 & 0 & \xi_7^7 & \xi_8^7 \\ 0 & 0 & \xi_3^8 & \xi_4^8 & 0 & 0 & \xi_7^8 & \xi_8^8 \end{array} \right), \quad M = \begin{pmatrix} \xi_3^3 & \xi_4^3 & \xi_7^3 & \xi_8^3 \\ \xi_3^4 & \xi_4^4 & \xi_7^4 & \xi_8^4 \\ \xi_3^7 & \xi_4^7 & \xi_7^7 & \xi_8^7 \\ \xi_3^8 & \xi_4^8 & \xi_7^8 & \xi_8^8 \end{pmatrix}. \tag{27}
$$

(ii) For $M, M' \in gl(4, \mathbb{R})$, there exists some $\Phi \in H$ such that $K(M') = \Phi K(M) \Phi^{-1}$ if and only if there exists $\left(\begin{smallmatrix} b_4^4 & b_8^4 \\ b_4^8 & b_8^8 \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ such that $M' = GMG^{-1}$, with

$$
G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_4^4 & 0 & b_8^4 \\ 0 & 0 & 1 & 0 \\ 0 & b_4^8 & 0 & b_8^8 \end{pmatrix} \in GL(4, \mathbb{R}). \tag{28}
$$

(iii) For $M, M' \in gl(4, \mathbb{R})$, there exists $\Psi \in \tau H$ such that $K(M') = \Psi K(M) \Psi^{-1}$ if and only if there exists $\Phi \in H$ such that $K(M') = \Phi K(M) \Phi^{-1}$.

Proof. (i) From lemma 3 and theorem 1(i), there exists $\Phi \in \left(SO(3) \times \mathbb{R}_+^*\right)^2 \subset H$ such that

$$
\Phi^{-1} J \Phi = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & \xi_5^1 & \xi_6^1 & \xi_7^1 & \xi_8^1 \\ -1 & 0 & 0 & 0 & \xi_5^2 & \xi_6^2 & \xi_7^2 & \xi_8^2 \\ 0 & 0 & \xi_3^3 & \xi_4^3 & \xi_5^3 & \xi_6^3 & \xi_7^3 & \xi_8^3 \\ 0 & 0 & \xi_3^4 & \xi_4^4 & \xi_5^3 & \xi_6^3 & \xi_7^4 & \xi_8^4 \\ \hline \xi_1^5 & \xi_2^5 & \xi_3^5 & \xi_4^5 & 0 & 1 & 0 & 0 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & -1 & 0 & 0 & 0 \\ \xi_1^7 & \xi_2^7 & \xi_3^7 & \xi_4^7 & 0 & 0 & \xi_7^7 & \xi_8^7 \\ \xi_1^8 & \xi_2^8 & \xi_3^8 & \xi_4^8 & 0 & 0 & \xi_7^8 & \xi_8^8 \end{array} \right). \tag{29}
$$

Hence we may suppose J of the form (29). The matrix $(\xi_j^i)_{1 \leqslant i \leqslant 4, 5 \leqslant j \leqslant 8}$ (resp. $(\xi_j^i)_{5 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 4}$) is the matrix of π_2^1 (resp. π_1^2) of lemma 3. Consider $\mathbf{u} = \pi_2^1 J_1^{(2)}, \mathbf{v} = \pi_2^1 J_2^{(2)}, \mathbf{w} = \pi_2^1 J_3^{(2)}, \mathbf{z} =$

$\pi^1_2 J^{(2)}_4$. From lemma 2 (ii) one has:

$$\begin{array}{rcl}
[\mathbf{u},\mathbf{v}] & = & 0, \\
[\mathbf{v},\mathbf{w}] & = & -\mathbf{v} + \xi^7_7 \mathbf{u}, \\
[\mathbf{u},\mathbf{w}] & = & -\mathbf{u} - \xi^7_7 \mathbf{v}, \\
[\mathbf{u},\mathbf{z}] & = & -\xi^7_8 \mathbf{v} \\
[\mathbf{v},\mathbf{z}] & = & \xi^7_8 \mathbf{u} \\
[\mathbf{w},\mathbf{z}] & = & 0
\end{array}$$

which implies $\mathbf{u} = \mathbf{v} = 0$. With the same reasoning for π^2_1, we get

$$J = \left(\begin{array}{cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & \xi^1_7 & \xi^1_8 \\
-1 & 0 & 0 & 0 & 0 & 0 & \xi^2_7 & \xi^2_8 \\
0 & 0 & \xi^3_3 & \xi^3_4 & 0 & 0 & \xi^3_7 & \xi^3_8 \\
0 & 0 & \xi^4_3 & \xi^4_4 & 0 & 0 & \xi^4_7 & \xi^4_8 \\
\hline
0 & 0 & \xi^5_3 & \xi^5_4 & 0 & 1 & 0 & 0 \\
0 & 0 & \xi^6_3 & \xi^6_4 & -1 & 0 & 0 & 0 \\
0 & 0 & \xi^7_3 & \xi^7_4 & 0 & 0 & \xi^7_7 & \xi^7_8 \\
0 & 0 & \xi^8_3 & \xi^8_4 & 0 & 0 & \xi^8_7 & \xi^8_8
\end{array} \right).$$

Now the torsion equations $17|3, 27|3$, $18|3, 28|3$, $35|7, 36|7$, $45|7, 46|7$, give the 4 Cramer systems $\xi^2_7\xi^3_3 - \xi^1_7 = 0$, $\xi^2_7 + \xi^3_3\xi^1_7 = 0$; $\xi^2_8\xi^3_3 - \xi^1_8 = 0$, $\xi^2_8 + \xi^3_3\xi^1_8 = 0$; $\xi^6_3\xi^7_7 - \xi^5_3 = 0$, $\xi^6_3 + \xi^7_7\xi^5_3 = 0$, $\xi^6_4\xi^7_7 - \xi^5_4 = 0$, $\xi^6_4 + \xi^7_7\xi^5_4 = 0$.

Hence $\xi^1_7 = \xi^2_7 = \xi^1_8 = \xi^2_8 = 0 = \xi^5_3 = \xi^6_3 = \xi^5_4 = \xi^6_4 = 0$. Then all torsion equations vanish ([3] `torsionu22.red`).

(ii) Suppose there exists

$$\Phi = \left(\begin{array}{cc|cc}
\Phi_1 & 0 & 0 & 0 \\
0 & b^4_4 & 0 & b^4_8 \\
\hline
0 & 0 & \Phi_4 & 0 \\
0 & b^8_4 & 0 & b^8_8
\end{array} \right) \in H, \quad \Phi_1, \Phi_4 \in SO(3), b^4_4 b^8_8 - b^4_8 b^8_4 \neq 0$$

such that $\Phi K(M) = K(M')\Phi$, with $M = \begin{pmatrix} \xi^3_3 & \xi^3_4 & \xi^3_7 & \xi^3_8 \\ \xi^4_3 & \xi^4_4 & \xi^4_7 & \xi^4_8 \\ \xi^7_3 & \xi^7_4 & \xi^7_7 & \xi^7_8 \\ \xi^8_3 & \xi^8_4 & \xi^8_7 & \xi^8_8 \end{pmatrix}$, $M' = \begin{pmatrix} {\xi'}^3_3 & {\xi'}^3_4 & {\xi'}^3_7 & {\xi'}^3_8 \\ {\xi'}^4_3 & {\xi'}^4_4 & {\xi'}^4_7 & {\xi'}^4_8 \\ {\xi'}^7_3 & {\xi'}^7_4 & {\xi'}^7_7 & {\xi'}^7_8 \\ {\xi'}^8_3 & {\xi'}^8_4 & {\xi'}^8_7 & {\xi'}^8_8 \end{pmatrix}$. One has $\Phi K(M) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A = \left(\begin{array}{ccc|c} & A_1 & & * \\ \hline * & * & * & * \end{array} \right)$, $D = \left(\begin{array}{ccc|c} & D_1 & & * \\ \hline * & * & * & * \end{array} \right)$, $A_1 = \Phi_1 \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^3_3 \end{smallmatrix}\right)$, $D_1 = \Phi_4 \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \xi^7_7 \end{smallmatrix}\right)$, and $K(M')\Phi = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$ with $A' = \left(\begin{array}{ccc|c} & A'_1 & & * \\ \hline * & * & * & * \end{array} \right)$, $D' = \left(\begin{array}{ccc|c} & D'_1 & & * \\ \hline * & * & * & * \end{array} \right)$, $A'_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & {\xi'}^3_3 \end{smallmatrix}\right) \Phi_1$, $D'_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & {\xi'}^7_7 \end{smallmatrix}\right) \Phi_4$. Hence ${\xi'}^3_3 = \xi^3_3$,

${\xi'}_7^7 = \xi_7^7$, and $\Phi_1 = \mathrm{diag}(R_1, 1)$, $\Phi_4 = \mathrm{diag}(R_4, 1)$, $R_1, R_4 \in SO(2)$. Now

$$\Phi = \begin{pmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_4^4 & 0 & 0 & b_8^4 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b_4^8 & 0 & 0 & b_8^8 \end{pmatrix},$$

and we can reindex the basis as the new basis $(J_1^{(1)}, J_2^{(1)}, J_1^{(2)}, J_2^{(2)}, J_3^{(1)}, J_4^{(1)}, J_3^{(2)}, J_4^{(2)})$, so that Φ, $K(M)$, $K(M')$ have respectives matrices in the new basis:

$$\Phi = \left(\begin{array}{cc|cccc} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4^4 & 0 & b_8^4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b_4^8 & 0 & b_8^8 \end{array}\right), \ K(M) = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & M \end{array}\right),$$

$$K(M') = \left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & M' \end{array}\right).$$

The conclusion follows.
(iii) One has $\tau K(M')\tau = K(\tau_1 M' \tau_1)$ with τ_1 the same as τ yet with 2×2 blocks: $\tau_1 = \left(\begin{smallmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{smallmatrix}\right)$. Now, let $\Psi = \tau\Phi \in \tau H$, $\Phi \in H$. Then $K(M') = \Psi K(M) \Psi^{-1}$ if and only if $\Phi K(M) \Phi^{-1} = \tau K(M')\tau = K(\tau_1 M' \tau_1)$ i.e. there exists $\left(\begin{smallmatrix} b_4^4 & b_8^4 \\ b_4^8 & b_8^8 \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ such that $\tau_1 M' \tau_1 = GMG^{-1}$ with G as in (28), i.e. $M' = (\tau_1 G) M (\tau_1 G)^{-1} = G_1 M G_1^{-1}$ with $G_1 = \tau_1 G = \left(\begin{smallmatrix} 1&0&0&0 \\ 0&b_8^8&0&b_4^8 \\ 0&0&1&0 \\ 0&b_8^4&0&b_4^4 \end{smallmatrix}\right)$ which is simply the matrix corresponding to $\left(\begin{smallmatrix} b_8^8 & b_4^8 \\ b_8^4 & b_4^4 \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ in the formula (28). □

Corollary 7. *Any $J \in \mathfrak{X}_{\mathfrak{u}(2)\oplus\mathfrak{u}(2)}$ is equivalent under some member of $\left(SO(3) \times \mathbb{R}_+^*\right)^2$ to $K(M)$ in (27) with $M \in GL(4, \mathbb{R})$, $M^2 = -I$. $K(M)$, $K(M')$ are equivalent if and only if there exists some $\left(\begin{smallmatrix} b_4^4 & b_8^4 \\ b_4^8 & b_8^8 \end{smallmatrix}\right) \in GL(2, \mathbb{R})$ such that $M' = GMG^{-1}$ with G as in (28).*

Proof. Follows readily from theorem 4. □

7 $U(2)^N$.

The results of lemma 6, theorem 4 and corollary 7 generalize in the following way.

Lemma 7. *For any $N \in \mathbb{N}^*$, $Aut\,(\mathfrak{u}(2))^N = H_N \cup \left(\bigcup_{\sigma\in\Sigma} \tau_\sigma\, H_N\right)$ (disjoint reunion) where:*

- $H_N = \left\{(U^i_j)_{1\leqslant i,j\leqslant N}; U^i_i = \begin{pmatrix}\Phi_i & 0\\ 0 & b^i_i\end{pmatrix}, U^i_j = \begin{pmatrix}0 & 0\\ 0 & b^j_i\end{pmatrix} (i \neq j), \Phi_i \in SO(3), \det(b^j_i) \neq 0\right\}$;
- Σ *is the set of circular permutations of* $\{1, \cdots, N\}$ *having no fixed point, and* $\tau_\sigma = (T^i_j)_{1\leqslant i,j\leqslant N}$ *with the* T^i_j*s the* 4×4 *blocks* $T^i_j = \delta_{i,\sigma(j)}\, I$ *(*I *the* 4×4 *identity and* $\delta_{k,\ell}$ *the Kronecker symbol).*

Theorem 5. *Let* $J : \mathfrak{u}(2)^N \to \mathfrak{u}(2)^N$. J *has zero torsion if and only if there exists* $\Phi \in \left(SO(3) \times \mathbb{R}^*_+\right)^N \subset H_N$ *and* $M = (M^i_j)_{1\leqslant i,j\leqslant N} \in gl(2N, \mathbb{R})$, $M^i_j = \begin{pmatrix}\xi^{4i-1}_{4j-1} & \xi^{4i-1}_{4j}\\ \xi^{4i}_{4j-1} & \xi^{4i}_{4j}\end{pmatrix}$, *such that* $\Phi^{-1} J \Phi = K(M)$ *with*

$$K(M) = (K^i_j(M))_{1\leqslant i,j\leqslant N} \tag{30}$$

and the $K^i_j(M)$*s the following* 4×4 *blocks:*

$$K^i_i(M) = \left(\begin{array}{cc|c} 0 & 1 & 0\\ -1 & 0 & 0\\ \hline 0 & 0 & M^i_i\end{array}\right) \quad (1 \leqslant i \leqslant N),$$

$$K^i_j(M) = \left(\begin{array}{cc|c} 0 & 0 & 0\\ 0 & 0 & 0\\ \hline 0 & 0 & M^i_j\end{array}\right) \quad (1 \leqslant i, j \leqslant N,\ i \neq j).$$

(Here we used that the analogs of $17|3, 27|3\ 18|3, 28|3\ 35|7, 36|7\ 45|7, 46|7$ at the end of (i) in the proof of Theorem 4 are resp. , with $i < j$,

$$\begin{aligned}
4i-3, 4j-1|4i-1: &\quad \xi^{4i-3}_{4j-1} - \xi^{4i-1}_{4i-1}\xi^{4i-2}_{4j-1} = 0\\
4i-2, 4j-1|4i-1: &\quad \xi^{4i-2}_{4j-1} + \xi^{4i-1}_{4i-1}\xi^{4i-3}_{4j-1} = 0\\
4i-3, 4j|4i-1: &\quad \xi^{4i-3}_{4j} - \xi^{4i-1}_{4i-1}\xi^{4i-2}_{4j} = 0\\
4i-2, 4j|4i-1: &\quad \xi^{4i-2}_{4j} + \xi^{4i-1}_{4i-1}\xi^{4i-3}_{4j} = 0\\
4i-1, 4j-3|4j-1: &\quad \xi^{4j-3}_{4i-1} - \xi^{4j-1}_{4j-1}\xi^{4j-2}_{4i-1} = 0\\
4i-1, 4j-2|4j-1: &\quad \xi^{4j-2}_{4i-1} + \xi^{4j-1}_{4j-1}\xi^{4j-3}_{4i-1} = 0\\
4i, 4j-3|4j-1: &\quad \xi^{4j-3}_{4i} - \xi^{4j-1}_{4j-1}\xi^{4j-2}_{4i} = 0\\
4i, 4j-2|4j-1: &\quad \xi^{4j-2}_{4i} + \xi^{4j-1}_{4j-1}\xi^{4j-3}_{4i} = 0
\end{aligned}$$

and give $\xi^{4i-3}_{4j-1} = \xi^{4i-2}_{4j-1} = \xi^{4i-3}_{4j} = \xi^{4i-2}_{4j} = \xi^{4j-3}_{4i-1} = \xi^{4j-2}_{4i-1} = \xi^{4j-3}_{4i} = \xi^{4j-2}_{4i} = 0$. Then all torsion equations vanish.)

Corollary 8. *Any* $J \in \mathfrak{X}_{\mathfrak{u}(2)^N}$ *is equivalent under some member of* $\left(SO(3) \times \mathbb{R}^*_+\right)^N$ *to* $K(M)$ *in (30) with* $M \in GL(2N, \mathbb{R})$, $M^2 = -I$. $K(M)$, $K(M')$ *are equivalent if and only if there exists some* $\left(b^{4i}_{4j}\right)_{1\leqslant i,j\leqslant N} \in GL(N, \mathbb{R})$ *such that* $M' = GMG^{-1}$ *with* $G = (G^i_j(M))_{1\leqslant i,j\leqslant N}$, $G^i_i = \begin{pmatrix}1 & 0\\ 0 & b^{4i}_{4i}\end{pmatrix}$, $G^i_j = \begin{pmatrix}0 & 0\\ 0 & b^{4i}_{4j}\end{pmatrix}$ $(i \neq j)$.

Remark 7. The closed set $\mathcal{R} = \{M \in GL(2N, \mathbb{R}); M^2 = -I\}$ is comprised of the conjugates of $\mathcal{T} = \begin{pmatrix}0 & -I_N\\ I_N & 0\end{pmatrix}$ (I_N the $N \times N$ identity) under the action of $GL(2N, \mathbb{R})$. Hence it is a

$2N^2$-dimensional submanifold of $\mathbb{R}^{4N^2}$ with a diffeomorphism $\chi : GL(2N,\mathbb{R}) / \mathcal{S} \to \mathcal{R}$, $\mathcal{S} = \left\{Q = \left(\begin{smallmatrix} R & -S \\ S & R \end{smallmatrix}\right) ; R, S \in GL(N,\mathbb{R}), \det Q \neq 0\right\}$ the stabilizer of $\mathcal{T}$, and χ defined by $\chi\,[P] = P\mathcal{T}P^{-1}$ for $[P]$ the class mod $\mathcal{S}$ of $P \in GL(2N,\mathbb{R})$. For $N=2$, $\chi\left[\left(\begin{smallmatrix} -\eta & 0 \\ \xi & 1 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} \xi & \eta \\ -\frac{1+\xi^2}{\eta} & -\xi \end{smallmatrix}\right)$, $(\xi,\eta) \in \mathbb{R} \times \mathbb{R}^*$. For general N and for $G \in GL(2N,\mathbb{R}), M = \chi\,[P]\,, M' = \chi\,[P'] \in \mathcal{R}$, $GMG^{-1} = \chi\,[GP]$ and the condition $M' = GMG^{-1}$ reads $[P'] = [GP]$.

8 Local chart and a representation for $(U(2), J(\xi))$.

8.1 Local chart.

For any fixed $\xi \in \mathbb{R}$, denote simply J the complex structure $J(\xi)$ on $\mathfrak{u}(2)$ and by G the group $U(2)$ endowed with the left invariant structure of complex manifold defined by J. For any open subset $V \subset U(2)$, the space $H_{\mathbb{C}}(V)$ of complex valued holomorphic functions on V (considered here as a subset of G) is comprised of all complex smooth functions f on V which are annihilated by all $\tilde{X}_j^- = X_j + iJX_j$ $(1 \leqslant j \leqslant 4)$ with $(X_j)_{1\leqslant j\leqslant 4}$ (resp. (JX_j)) the left invariant vector fields associated to the basis $(J_j)_{1\leqslant j\leqslant 4}$ of $\mathfrak{u}(2)$ (resp. to $(J\,J_j)$). One has $\tilde{X}_1^- = X_1 - iX_2$, $\tilde{X}_2^- = i\tilde{X}_1^-$, $\tilde{X}_4^- = iX_3 + (1-i\xi)X_4$, $\tilde{X}_3^- = -i(1+i\xi)\tilde{X}_4^-$, hence

$$H_{\mathbb{C}}(V) = \{f \in C^\infty(V) \; ; \; \tilde{X}_1^- f = \tilde{X}_4^- f = 0\}. \tag{31}$$

As is known, the map $\mathbb{S}^1 \times SU(2) \to U(2)$ defined by

$$(\zeta, A) \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} A$$

is a diffeomorphism of manifolds (not of groups). Introducing Euler angles as coordinates in the open subset $\Omega = SU(2) \setminus \left(e^{\mathbb{R}J_1} \cup e^{\pi J_3} e^{\mathbb{R}J_1}\right)$ of $SU(2)$, one gets the coordinates $(s, \theta, \varphi, \psi)$ in the open subset

$$V = \left(\mathbb{S}^1 \setminus \{-1\}\right) \times \Omega \tag{32}$$

such that u defined by

$$u(s,\theta,\varphi,\psi) = \begin{pmatrix} e^{is} & 0 \\ 0 & 1 \end{pmatrix} e^{\varphi J_3} e^{\theta J_1} e^{\psi J_3} = \begin{pmatrix} e^{is} e^{i\frac{\varphi+\psi}{2}} \cos\frac{\theta}{2} & ie^{is} e^{i\frac{\varphi-\psi}{2}} \sin\frac{\theta}{2} \\ ie^{-i\frac{\varphi-\psi}{2}} \sin\frac{\theta}{2} & e^{-i\frac{\varphi+\psi}{2}} \cos\frac{\theta}{2} \end{pmatrix} \tag{33}$$

is a diffeomorphism of $]-\pi,\pi[\times]0,\pi[\times]0,2\pi[\times]-2\pi,2\pi[$ on V. Then one gets on V (see e.g. [12], p.141):

$$\begin{aligned}
X_1 &= \cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\varphi} - \cot\theta \sin\psi \frac{\partial}{\partial\psi} \\
X_2 &= -\sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\varphi} - \cot\theta \cos\psi \frac{\partial}{\partial\psi} \\
X_3 &= \frac{\partial}{\partial\psi} \\
X_4 &= \frac{\partial}{\partial s} - \frac{\partial}{\partial\varphi}
\end{aligned}$$

Hence $f \in C^\infty(V)$ is in $H_{\mathbb{C}}(V)$ if and only if it satisfies the 2 equations

$$\begin{aligned} i\sin\theta\,\frac{\partial f}{\partial\theta} + \frac{\partial f}{\partial\varphi} - \cos\theta\,\frac{\partial f}{\partial\psi} &= 0 && (34)\\ i\,\frac{\partial f}{\partial\psi} + (1-i\xi)\left(\frac{\partial f}{\partial s} - \frac{\partial f}{\partial\varphi}\right) &= 0 && (35)\end{aligned}$$

The 2 functions

$$\begin{aligned} w^1 &= e^{i(s+\varphi)}\cot\frac{\theta}{2} && (36)\\ w^2 &= e^{(1+i\xi)\frac{s}{2(1+\xi^2)}}\,e^{i\frac{\psi}{2}}\sqrt{\sin\theta} && (37)\end{aligned}$$

are holomorphic on V. Let $F : V \to \mathbb{C}^2$ defined by $F = (w^1, w^2)$. It is easily seen that F is injective, with jacobian $-\frac{1}{4(1+\xi^2)}\,e^{\frac{s}{1+\xi^2}}\,(\cot\frac{\theta}{2})^2 \neq 0$, hence F is a biholomorphic bijection of V onto an open subset $F(V)$ of $\mathbb{C}^2$, i.e. (V, F) is a chart of G. $F(V)$ is the set of those $(w^1, w^2) \in \mathbb{C}^2$ satisfying the following conditions, where $r_1 = |w^1|, r_2 = |w^2|$ and $\omega(r_1, r_2) = \log r_2 - \frac{1}{2}\log\frac{2r_1}{1+r_1^2}$: $r_1 r_2 \neq 0$, , $\sqrt{\frac{2r_1}{1+r_1^2}}\,e^{-\frac{\pi}{2(1+\xi^2)}} < r_2 < \sqrt{\frac{2r_1}{1+r_1^2}}\,e^{\frac{\pi}{2(1+\xi^2)}}$, $\arg w_1 \not\equiv 2(1+\xi^2)\omega(r_1, r_2) \mod 2\pi$, $\arg w_2 \not\equiv \xi\omega(r_1, r_2) + \pi \mod 2\pi$. For example, if $\xi = 0$,

$$V = \bigcup_{r_1>0}\ \bigcup_{e^{-\frac{\pi}{2}}y(r_1)<r_2<e^{\frac{\pi}{2}}y(r_1)} \left(\left(\mathcal{C}^{(1)}_{r_1}\setminus\{\arg\ \equiv 2\omega(r_1,r_2)\}\right)\times\left(\mathcal{C}^{(2)}_{r_2}\setminus\{\arg\ \equiv\pi\}\right)\right)$$

where $\mathcal{C}^j_{r_j}$ $(j = 1, 2)$ is the circle with radius r_j in the w^j-plane and $y(x) = \sqrt{\frac{2x}{1+x^2}}$ $(x > 0)$.

8.2 A representation on a space of holomorphic functions.

As $U(2)$ is compact, there are no nonconstant holomorphic functions on the whole of $U(2)$. Instead, we consider the space $H_{\mathbb{C}}(V)$ of holomorphic functions on the open subset V (32), and we compute (as kind of substitute for the regular representation) the representation λ of the Lie algebra $\mathfrak{u}(2)$ we get by Lie derivatives on $H_{\mathbb{C}}(V)$. First, note that for any $x = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V$ as in (33), the complex coordinates w^1, w^2 of x (36,37) satisfy:

$$\begin{aligned} w^1 &= -ie^{-is}\frac{a}{\bar b} && (38)\\ (w^2)^2 &= 2ia\bar b\,e^{s\frac{1+i\xi}{1+\xi^2}}. && (39)\end{aligned}$$

Then one gets for the complex coordinates $w^1_{e^{-tJ_1}x}, w^2_{e^{-tJ_1}x}$ of $e^{-tJ_1}x$ $(x \in V,\ t \in \mathbb{R}$ sufficiently small):

$$\begin{aligned} w^1_{e^{-tJ_1}x} &= \frac{1 + w^1\cot\frac{t}{2}}{\cot\frac{t}{2} - w^1}\\ (w^2_{e^{-tJ_1}x})^2 &= (w^2)^2\left(\cos t + \frac{\sin t}{2}\,\frac{1-(w^1)^2}{w^1}\right).\end{aligned}$$

Whence for any $f \in H_{\mathbb{C}}(V)$, denoting $J_1 f$ instead of $\lambda(J_1)f$:

$$(J_1 f)\,(w^1, w^2) = \left[\frac{d}{dt} f(w^1_{e^{-tJ_1}x}, w^2_{e^{-tJ_1}x})\right]_{t=0} = \frac{1+(w^1)^2}{2}\,\frac{\partial f}{\partial w^1} + \frac{w^2\left(1-(w^1)^2\right)}{4w^1}\,\frac{\partial f}{\partial w^2}.$$

In the same way,

$$\begin{aligned}
w^1_{e^{-tJ_2}x} &= -i\,\frac{i\sin\frac{t}{2} + w^1\,\cos\frac{t}{2}}{-i\cos\frac{t}{2} + w^1\,\sin\frac{t}{2}} \\
(w^2_{e^{-tJ_2}x})^2 &= \left(w^2\right)^2\left(\cos t + i\,\frac{\sin t}{2}\,\frac{1+\left(w^1\right)^2}{w^1}\right) \\
(J_2 f)\,(w^1, w^2) &= \frac{i(1-(w^1)^2)}{2}\,\frac{\partial f}{\partial w^1} + \frac{iw^2\left(1+(w^1)^2\right)}{4w^1}\,\frac{\partial f}{\partial w^2} \\
w^1_{e^{-tJ_3}x} &= e^{-it}w^1 \\
(w^2_{e^{-tJ_3}x})^2 &= \left(w^2\right)^2 \\
(J_3 f)\,(w^1, w^2) &= -iw^1\,\frac{\partial f}{\partial w^1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
w^1_{e^{-tJ_4}x} &= w^1 \\
(w^2_{e^{-tJ_4}x})^2 &= \left(w^2\right)^2 e^{-t\,\frac{1+i\xi}{1+\xi^2}} \\
(J_4 f)\,(w^1, w^2) &= -\frac{1+i\xi}{1+\xi^2}\,w^2\,\frac{\partial f}{\partial w^2}.
\end{aligned}$$

In the complexification $\mathfrak{sl}(2) \oplus \mathbb{C}J_4$ of $\mathfrak{u}(2)$, introduce as usual

$$H_\pm = iJ_1 \mp J_2,\;\; H_3 = iJ_3$$

so that

$$[H_3, H_\pm] = \pm H_\pm,\;\; [H_+, H_-] = 2H_3.$$

Then, extending the representation λ to $\mathfrak{sl}(2) \oplus \mathbb{C}J_4$, one has, with $H_4 = -(1-i\xi)J_4$:

$$(H_+ f)\,(w^1, w^2) = i\left((w^1)^2\frac{\partial f}{\partial w^1} - \frac{1}{2}\,w^1 w^2\,\frac{\partial f}{\partial w^2}\right) \tag{40}$$

$$(H_- f)\,(w^1, w^2) = i\left(\frac{\partial f}{\partial w^1} + \frac{w^2}{2w^1}\,\frac{\partial f}{\partial w^2}\right) \tag{41}$$

$$(H_3 f)\,(w^1, w^2) = w^1\,\frac{\partial f}{\partial w^1}. \tag{42}$$

$$(H_4 f)\,(w^1, w^2) = w^2\,\frac{\partial f}{\partial w^2}. \tag{43}$$

8.3 A subrepresentation.

We restrict λ to $H(\mathbb{C}^* \times \mathbb{C}^*)$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$), and denote $\varphi_{p,q}$ the function $\varphi_{p,q}(w^1, w^2) = (w^1)^p (w^2)^q$ for $p, q \in \mathbb{Z}$. The system $(\varphi_{p,q})_{p,q\in\mathbb{Z}}$ is total in $H(\mathbb{C}^* \times \mathbb{C}^*)$, and one has:

$$\begin{aligned} H_+ \varphi_{p,q} &= i\left(p - \frac{q}{2}\right) \varphi_{p+1,q} && (44) \\ H_- \varphi_{p,q} &= i\left(p + \frac{q}{2}\right) \varphi_{p-1,q} && (45) \\ H_3 \varphi_{p,q} &= p\, \varphi_{p,q} && (46) \\ H_4 \varphi_{p,q} &= q\, \varphi_{p,q} && (47) \end{aligned}$$

For any $q \in \mathbb{Z}$, the subspace $\mathcal{H}_q$ of functions of the form $(w^2)^q g(w^1)$, $g \in H(\mathbb{C}^*)$, is a closed invariant subspace of $H(\mathbb{C}^* \times \mathbb{C}^*)$, and $H(\mathbb{C}^* \times \mathbb{C}^*)$ is the closure of $\bigoplus_{q\in\mathbb{Z}} \mathcal{H}_q$.

8.4 A lemma.

Lemma 8. *Let $\mathcal{E} = H(\mathbb{C}^*)$ the Fréchet space of holomorphic functions of the complex variable z on $\mathbb{C}^*$. Let $\mathcal{F}$ be any closed vector subspace of $\mathcal{E}$ that is invariant by the operator $z\frac{d}{dz}$. Let $f \in \mathcal{F}$ and $f(z) = \sum_{p=-\infty}^{+\infty} c_p z^p$ its Laurent expansion in $\mathbb{C}^*$. If for some $p \in \mathbb{Z}, c_p \neq 0$, then the function $z \mapsto z^p$ belongs to $\mathcal{F}$.*

Proof. We show first that the function $z \mapsto f(e^{i\theta} z)$ belongs to $\mathcal{F}$ $\forall \theta \in \mathbb{R}$ $\forall f \in \mathcal{F}$. Let $f \in \mathcal{F}$ and $f(z) = \sum_{p=-\infty}^{+\infty} c_p z^p$ its Laurent expansion in $\mathbb{C}^*$. Since it is uniformly and absolutely convergent on compact subsets of $\mathbb{C}^*$, and since the operator $H = z\frac{d}{dz}$ is continuous on $\mathcal{E}$,

$$\frac{(i\theta)^k}{k!}\,(H^k f)(z) = \sum_{p=-\infty}^{\infty} c_p \frac{(i\theta p)^k}{k!}\, z^p, \quad \forall k \in \mathbb{N}, \forall \theta \in \mathbb{R}, \forall z \in \mathbb{C}^*.$$

On the other hand, for any fixed $\theta \in \mathbb{R}$, the double series

$$\sum_{k=0}^{+\infty} \sum_{p=-\infty}^{+\infty} c_p \frac{(i\theta p)^k}{k!} z^p$$

is absolutely and uniformly summable in the annulus $A(r, R)$ for any $0 < r < R < +\infty$ since

$$\sum_{k=0}^{+\infty} \sum_{p=-\infty}^{+\infty} |c_p| \frac{(|\theta||p|)^k}{k!} |z|^p \leqslant \sum_{p<0} |c_p| (e^{-|\theta|} r)^p + \sum_{p\geqslant 0} |c_p| (e^{|\theta|} R)^p < +\infty.$$

From the associativity theorem for summable families,

$$f(e^{i\theta} z) = \sum_{k=0}^{+\infty} \frac{(i\theta)^k}{k!}\,(H^k f)(z)$$

with the series uniformly and absolutely convergent on compact subsets of $\mathbb{C}^*$. The conclusion follows, since $H^k f \in \mathcal{F}$ $\forall k$. Now we use the same trick as in ([5], p. 14). For any $z \in \mathbb{C}^*$, denote f_z the periodic function on $\mathbb{R} : \theta \mapsto f(e^{i\theta}z)$. Its Fourier expansion is $f(e^{i\theta}z) = \sum_{p=-\infty}^{+\infty} \tilde{c}_p(z)e^{ip\theta}$ where

$$\tilde{c}_p(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)e^{-ip\theta}d\theta.$$

The function $z \mapsto \tilde{c}_p(z)$ belongs to $\mathcal{F}$ as the right-hand side is a limit in $\mathcal{E}$ of linear combinations of functions $z \mapsto f(e^{i\theta}z)$. But with the Laurent expansion of f one gets $f(e^{i\theta}z) = \sum_{p=-\infty}^{+\infty} c_p z^p e^{ip\theta}$. For any z, that series is a trigonometric series that converges uniformly on $\mathbb{R}$ hence it coincides with the Fourier series of f_z and $\tilde{c}_p(z) = c_p z^p$ $\forall p \in \mathbb{Z}$. Hence if for some $p \in \mathbb{Z}, c_p \neq 0$, then the function $z \mapsto z^p$ belongs to $\mathcal{F}$. $\square$

8.5 A closer look to the subrepresentation.

Introduce the Casimir $C = H_+H_- + (H_3)^2 - H_3$. On $\mathcal{H}_q$, $C = u(u+1)$ with $u = \frac{q}{2}$. Now, we distinguish cases. We use both the notations $\uparrow_u^q, D^q(2k)$, etc. of [8] (Th. 2.3) for representations of $\mathfrak{u}(2)$ and usual notations of e.g. [9] (7.3) $\uparrow_u, D^{(k)}$, etc. for representations of $\mathfrak{sl}(2)$. One has $\uparrow_u^q = \uparrow_u \otimes q$, $D^q(2k) = D^{(k)} \otimes q$ etc.

Case 1: $q = -2k$, $k \in \mathbb{N} \setminus \{0\}$. Then from (44), (45), the closed subspace $\mathcal{H}_q^\uparrow$ (resp. $\mathcal{H}_q^\downarrow$) generated by $\{\varphi_{k+n,q}, n \in \mathbb{N}\}$, (resp. $\{\varphi_{-k-n,q}, n \in \mathbb{N}\}$), which is comprised of the functions $(w^2)^{-2k}(w^1)^k g(w^1)$ (resp. $(w^2)^{-2k}(w^1)^{-k} g(\frac{1}{w^1})$), $g \in H(\mathbb{C})$, is invariant and topologically irreducible from lemma 8. $\mathcal{H}_q^\uparrow = \uparrow_{-k}^q = \uparrow_{-k} \otimes q$, $\mathcal{H}_q^\downarrow = \downarrow_{-k}^q = \downarrow_{-k} \otimes q$. $\mathcal{H}_q$ is indecomposable and $\mathcal{H}_q / (\mathcal{H}_q^\uparrow \oplus \mathcal{H}_q^\downarrow)$ is top. irreducible and equal to $D^q(2(k-1)) = D^{(k-1)} \otimes q$, i.e. $\mathcal{H}_q$ is a nontrivial extension of $D^q(2(k-1))$ by $\uparrow_{-k}^q \oplus \downarrow_{-k}^q$.

Case 2: $q = 2k$, $k \in \mathbb{N}$. Then the closed subspace $\mathcal{H}_q^D$ generated by $\{\varphi_{-k+n,q}, n \in \mathbb{N}, 0 \leqslant n \leqslant 2k\}$, which is comprised of the functions $(w^2)^{2k}(w^1)^{-k}P(w^1)$, $P \in \mathbb{C}[w^1], \deg P \leqslant 2k$, is invariant and topologically irreducible from lemma 8, and $\mathcal{H}_q^D = D^q(2k) = D^{(k)} \otimes q$. There are exactly 2 closed invariant (nontrivial) subspaces containing $\mathcal{H}_q^D$. Each one is indecomposable, with top. irreducible quotient by $\mathcal{H}_q^D$ equal respectively to $\uparrow_{-k-1}^q$ or $\downarrow_{-k-1}^q$.

Case 3: $q \notin 2\mathbb{Z}$. In that case $\mathcal{H}_q = D^q(u, 0)$ is top. irreducible.

We see that λ is quite different from the regular representation, since the differentials of the representation in the unitary dual of $U(2)$ are $D^{(\ell)} \otimes m$, $2\ell \in \mathbb{N}, m \in \mathbb{Z}$, with $2\ell + m$ even ([1], p. 87).

9 Chart for $(SU(2) \times SU(2), J(\xi, \eta))$.

In this last section, we compute an holomorphic chart for $J(\xi, \eta)$, $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^*$, in the open subset $W = \Omega \times \Omega$ of $SU(2) \times SU(2)$ with Euler angles coordinates $(\theta_1, \phi_1, \psi_1, \theta_2, \phi_2, \psi_2)$. The space $H_{\mathbb{C}}(W)$ of complex valued holomorphic functions on W is comprised of all complex smooth functions f on W which are annihilated by all

$$\tilde{X}_j^{(k)^-} = X_j^{(k)} + iJX_j^{(k)}, \quad 1 \leqslant j \leqslant 3, \quad 1 \leqslant k \leqslant 2, \tag{48}$$

$(X_j^{(k)})$ the left invariant vector fields associated to the basis $(J_1^{(1)}, J_2^{(1)}, J_3^{(1)}, J_1^{(2)}, J_2^{(2)}, J_3^{(2)})$ of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. One has $\tilde{X}_1^{(k)^-} = X_1^{(k)} - iX_2^{(k)}$, $\tilde{X}_2^{(k)^-} = i\tilde{X}_1^{(k)^-}$ for $(k = 1, 2)$, and

$\tilde{X}_3^{(2)\,-} = i\eta X_3^{(1)} + (1-i\xi)X_3^{(2)}$, $\tilde{X}_3^{(1)\,-} = -i\frac{1+i\xi}{\eta}\tilde{X}_3^{(2)\,-}$. For $k=1,2$,

$$\begin{aligned}
X_1^{(k)} &= \cos\psi_k \frac{\partial}{\partial\theta_k} + \frac{\sin\psi_k}{\sin\theta_k}\frac{\partial}{\partial\varphi_k} - \cot\theta_k \sin\psi_k \frac{\partial}{\partial\psi_k} \\
X_2^{(k)} &= -\sin\psi_k \frac{\partial}{\partial\theta_k} + \frac{\cos\psi_k}{\sin\theta_k}\frac{\partial}{\partial\varphi_k} - \cot\theta_k \cos\psi_k \frac{\partial}{\partial\psi_k} \\
X_3^{(k)} &= \frac{\partial}{\partial\psi_k}\,.
\end{aligned}$$

Hence $f \in C^\infty(W)$ is in $H_{\mathbb{C}}(W)$ if and only if it satisfies the 3 equations

$$i\sin\theta_1 \frac{\partial f}{\partial\theta_1} + \frac{\partial f}{\partial\varphi_1} - \cos\theta_1 \frac{\partial f}{\partial\psi_1} = 0 \tag{49}$$

$$i\sin\theta_2 \frac{\partial f}{\partial\theta_2} + \frac{\partial f}{\partial\varphi_2} - \cos\theta_2 \frac{\partial f}{\partial\psi_2} = 0 \tag{50}$$

$$i\eta \frac{\partial f}{\partial\psi_1} + (1-i\xi)\frac{\partial f}{\partial\psi_2} = 0\,. \tag{51}$$

The 3 functions

$$z^1 = e^{i\varphi_1}\cot\frac{\theta_1}{2} \tag{52}$$

$$z^2 = e^{i\varphi_2}\cot\frac{\theta_2}{2} \tag{53}$$

$$z^3 = e^{i\frac{\psi_1}{2}}\, e^{\frac{\eta(1+i\xi)}{1+\xi^2}\frac{\psi_2}{2}}\sqrt{\sin\theta_1}\sqrt{\sin\theta_2} \tag{54}$$

are holomorphic on W. Let $Z : W \to \mathbb{C}^3$ defined by $Z = (z^1, z^2, z^3)$. Z is injective, with jacobian $-\frac{\eta}{4(1+\xi^2)}\, e^{\frac{\eta}{1+\xi^2}\psi_2}(\cot\frac{\theta_1}{2})^2(\cot\frac{\theta_2}{2})^2 \neq 0$, hence Z is a biholomorphic bijection of W onto an open subset of $\mathbb{C}^3$, i.e. (W, Z) is a local chart for $SU(2)\times SU(2)$ equipped with the complex structure $J(\xi,\eta)$.

References

[1] T. Bröcker, T. Dieck *Representations of Compact Lie Groups*, Graduate texts in Math. #98, Springer, New York, 1985.

[2] J.-Y. Charbonnel, H. O. Khalgui, Classification des structures CR invariantes pour les groupes de Lie compacts, *J. of Lie Theory*, **14**, 2004, 165-198.

[3] `http://www.u-bourgogne.fr/monge/l.magnin/CSu2/CSu2index.html` or `http://math.u-bourgogne.fr/IMB/magnin/public_html/CSu2/CSu2index.html`

[4] N. A. Daurtseva, Invariant complex structures on $\mathbb{S}^3\times\mathbb{S}^3$, Electronic journal *Investigated in Russia*, 2004, 888-893.
English version `http://zhurnal.ape.relarn.ru/articles/2004/081e.pdf`
Russian version `http://zhurnal.ape.relarn.ru/articles/2004/081.pdf`

[5] S. Helgason, *Groups and geometric analysis (integral geometry, invariant differential operators and spherical functions)*, Academic Press, Orlando, 1984.

[6] L. Magnin, Complex structures on indecomposable 6-dimensional nilpotent real Lie algebras, *Int. J. Alg. Comput.*, **17**, #1, 2007, 77-113.

[7] L. Magnin, Left invariant complex structures on real 6-dimensional simply connected indecomposable nilpotent Lie groups, *Int. J. Alg. Comput.*, **17**, #1, 2007, 115-139.

[8] W. S. Miller Jr., *Lie theory and special functions*, Academic Press, New York, 1968.

[9] W. Miller Jr., *Symmetry groups and their applications*, Academic Press, New York, 1972.

[10] A. Newlander, L. Nirenberg, Complex analytic coordinates in almost complex manifolds, *Ann. Math.*, **65**, 1957, 391-404.

[11] T. Sasaki, Classification of left invariant complex structures on $GL(2,\mathbb{R})$ and $U(2)$, *Kumamoto J. Sci. (Math)*, **14**, 1981, 115-123.

[12] N. Ja. Vilenkin, *Fonctions spéciales et théorie de la représentation des groupes*, Dunod, Paris, 1969.