Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras

L. Magnin (magnin@u-bourgogne.fr)
Institut Mathématique de Bourgogne

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Abstract. For any complex 6-dimensional nilpotent Lie algebra $\mathfrak{g}$, we compute the strain of all indecomposable 7-dimensional nilpotent Lie algebras which contain $\mathfrak{g}$ by the adjoining a derivation method. We get a new determination of all 7-dimensional complex nilpotent Lie algebras, allowing to check earlier results (some contain errors), along with a cross table intertwining nilpotent 6- and 7-dimensional Lie algebras.

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1. Introduction.

Given a nilpotent Lie algebra (abbreviation: NLA) over $\mathbb{R}$ or $\mathbb{C}$, the unitary dual of the corresponding simply connected nilpotent Lie group is nicely described by the orbit method of Kirillov [12], [13], [6]. However very little is known on the description of the NLAs themselves. The classification of NLAs of dimension $\leqslant 6$ goes back to Dixmier [7] (dim $\leqslant 5$), Morosov [22] and Vergne [29] (dim 6). Here, there are finitely many (isomorphy classes of) NLAs. Moreover there is a 1:1 correspondence between (equivalence classes of) weight systems with respect to a maximal torus and (isomorphy classes of) NLAs. Non isomorphic NLAs of dimension $\leqslant 6$ also have different adjoint cohomologies, even though their trivial cohomologies may be equal.

In dimension 7, the first classification appeared in 1964 based on Morosov's method [26]. Several classifications appeared later based on different approaches [1],[9],[25],[27]. However, due to the lack of a good invariant and the complexity of the calculations involved, the completeness issue of the lists was a recurrent problem.

Dimension 7 is much of a change: continuous 1-parameter series enter, the so-called characteristically nilpotent Lie algebras appear, the correspondence between (equivalence classes of) weight systems and (isomorphy classes of) NLAs is no longer 1:1, and up to 14 nonisomorphic NLAs (of which 2 continuous series) may share the same trivial cohomology. Fortunately, adjoint cohomology still separates most algebras [18].

The weight systems for complex NLAs of dimension ≤ 7 have been computed by Carles [5]. He deduced a new classification of complex 7-dimensional NLAs. However, no proofs appeared in [5]. That classification was used in [18].

The method of adjoining a derivation, which was first introduced in [4] to get conditions for algebraicity and rigidity, has been applied in [14] to the classification problem. Here, one introduces for any complex NLA $\mathfrak{g}$ and any nilpotent derivation $\delta \in \mathrm{Der}\,(\mathfrak{g})$ the semidirect product $\tilde{\mathfrak{g}}_\delta = \mathbb{C}\delta \oplus \mathfrak{g}$ i.e. the extension $0 \to \mathfrak{g} \to \tilde{\mathfrak{g}}_\delta \to \mathbb{C} \to 0$ with $\mathbb{C}$ acting on $\mathfrak{g}$ via δ.

The application to the 7-dimensional case was freezed by the amount of calculations involved, the difficulty being the separation into nonisomorphic types. In fact, the nilpotent $\mathrm{Aut}\,\mathfrak{g}$-orbits in the projective adjoint cohomology space $PH^1(\mathfrak{g})$ classify all $\tilde{\mathfrak{g}}_\delta$s which give *quasi-equivalent extensions*. However, a major unsolved question would be to define an object classifying all $\tilde{\mathfrak{g}}_\delta$s that are isomorphic as *Lie algebras*, i.e. to find conditions on δ, δ' for $\tilde{\mathfrak{g}}_\delta, \tilde{\mathfrak{g}}_{\delta'}$ to be isomorphic Lie algebras.

The recent progresses in computer technology and symbolic computation softwares have opened new perspectives. In [8], the Skjelbred-Sund method [28] has been applied with computer calculations to get a classification which essentially agrees with Seeley's one and which eliminates some minor errors.

In the present paper, for any given 6-dimensional complex NLA $\mathfrak{g}$, we apply the adjoining a derivation method along with the computer algebra system *Reduce* to determine (up to isomorphism) the set of all complex indecomposable 7-dimensional NLAs containing $\mathfrak{g}$, which we call the strain of $\mathfrak{g}$. Our results are a new determination of all 7-dimensional complex indecomposable NLAs, allowing to check earlier results (some contain errors), like the completeness of the list in [5] (some corrections needed), and a bijection between the (corrected) lists [5] and [27], along with a cross table intertwining nilpotent 6-dimensional NLAs and 7-dimensional indecomposable NLAs.

Recently, further work has been done in dimension 7. A.I. Ooms [23] determined the centers of the enveloping algebras of 7-dimensional nilpotent Lie algebras in the cases (64 cases) where the index of $\mathfrak{g}$ (i.e. the minimum of the dimensions of the stabilizers in $\mathfrak{g}$ of linear forms on $\mathfrak{g}$) is smaller or equal to the rank of $\mathfrak{g}$.

2. Preliminaries.

2.1. Historical background and labeling of the algebras.

Dixmier classified 5-dimensional real and complex NLAs [7]: there is (up to isomorphism) 6 indecomposable NLAs and he introduced historical notations $\mathfrak{g}_{5,j}, 1 \leqslant j \leqslant 6$. The 6-dimensional case was treated by Morosov [22], based on the maximal dimension of an abelian ideal. There are 22 indecomposable nonisomorphic real 6-dimensional NLAs in the Morozov classification, labeled $M1$-$M22$ [14]. Types $M14$ and $M18$ are split into $M14_{\pm 1}$ and $M18_{\pm 1}$. Over $\mathbb{C}$, types $M14$ and $M18$ are not split and types $M5$ and $M10$ do not appear since then $M5 \cong \mathfrak{n} \times \mathfrak{n}$ where $\mathfrak{n}$ denotes the 3-dimensional complex Heisenberg Lie algebra and $M10 \cong M8$. Hence there are 20 nonisomorphic indecomposable complex NLAs. Over $\mathbb{C}$, Vergne [29] also computed in a different way all 6-dimensional NLAs, and introduced historical notations $\mathfrak{g}_{6,j}, 1 \leqslant j \leqslant 20$ (These notations are not to be confused with those of [24], p.210-211, which refer to Morozov types.)

In [5], Carles computed all weight systems for complex NLAs of dimension ≤ 7, and deduced a new classification in dimension 7 with notations taking into account the rank and his labeling of the weight systems, in the spirit of the historical notations. However, for dimension ≤ 6, the historical notations have not be changed, though not related to the rank.

2.2. Carles' list.

As there are a few errors in [5], along with a few changes, with respect to the earlier commutation relations that were the one we used in [18], there is a point in summarizing the commutation relations for Carles' list in the form we used in [18]. There are 98 (non equivalent) weight systems for complex 7-dimensional indecomposable NLAs: 1 in rank 0, 24 in rank 1, 45 * in rank 2, 24 in rank 3 and 4 in rank 4. The indecomposable NLAs of dimension 7 are almost classified by their weight system with a few exceptions, and one gets 123 non isomorphic indecomposable NLAs of dimension 7, of which 6 continuous 1-parameter series (each continuous series counts as one algebra). All commutation relations are given in a basis $(x_k)_{1\leqslant k\leqslant 7}$ which diagonalizes a maximal torus. To save space, we use a short notation: for example $\mathfrak{g}_{7,0.4(\lambda)}$ $(\lambda\in\mathbb{C})$: $12|3, 13|4, 14|(6+\lambda\cdot 7), 15|7, 16|7, 23|5, 24|7, 25|6, 35|7$. stands for $\mathfrak{g}_{7,0.4(\lambda)}$ $(\lambda\in\mathbb{C})$: $[x_1,x_2]=x_3$, $[x_1,x_3]=x_4$, $[x_1,x_4]=x_6+\lambda x_7$, $[x_1,x_5]=x_7$, $[x_1,x_6]=x_7$, $[x_2,x_3]=x_5$, $[x_2,x_4]=x_7$, $[x_2,x_5]=x_6$, $[x_3,x_5]=x_7$. Also $45|-7$ stands for $[x_4,x_5]=-x_7$ and $45|(1-\lambda)\cdot 7$ stands for $[x_4,x_5]=(1-\lambda)x_7$.

Rank 0 (characteristically nilpotent) : 8 algebras of which 1 continuous 1-parameter series.

$\mathfrak{g}_{7,0.1}$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|6, 24|7, 25|7, 34|-7$. $\mathfrak{g}_{7,0.2}$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|(5+7), 24|6, 25|7$. $\mathfrak{g}_{7,0.3}$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|(6+7), 24|7$. $\mathfrak{g}_{7,0.4(\lambda)}$ $(\lambda\in\mathbb{C})$: $12|3, 13|4, 14|(6+\lambda\cdot 7), 15|7, 16|7, 23|5, 24|7, 25|6, 35|7$. $\mathfrak{g}_{7,0.5}$: $12|3, 13|4, 14|(6+7), 16|7, 23|5, 25|6, 35|7$. $\mathfrak{g}_{7,0.6}$: $12|3, 13|4, 14|7, 15|6, 16|7, 23|5, 24|6, 25|7, 34|7$. $\mathfrak{g}_{7,0.7}$: $12|3, 13|4, 14|7, 15|7, 16|7, 23|5, 24|7, 25|6, 35|7$. $\mathfrak{g}_{7,0.8}$: $12|4, 13|7, 14|5, 15|6, 23|6, 24|6, 26|7, 45|-7$.

Rank 1 with a null weight : 4 algebras.

$\mathfrak{g}_{7,1.01(i)}$ † : $12|4, 14|5, 15|6, 16|7, 23|(5+7), 34|-6, 35|-7$. $\mathfrak{g}_{7,1.01(ii)}$ † : $12|4, 14|5, 15|6, 16|7, 23|(6+7), 34|-7$. $\mathfrak{g}_{7,1.02}$: $12|3, 13|4, 15|6, 23|5, 24|6, 25|7, 26|7, 35|-7$. $\mathfrak{g}_{7,1.03}$: $12|3, 13|4, 14|5, 16|7, 23|6, 24|7, 25|7, 34|-7$.

Rank 1 without a null weight : 33 algebras of which 3 continuous 1-parameter series.

$\mathfrak{g}_{7,1.1(i_\lambda)}$ $(\lambda\in\mathbb{C})$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|5, 24|6, 25|\lambda\cdot 7, 34|(1-\lambda)\cdot 7$. $\mathfrak{g}_{7,1.1(ii)}$: $12|3, 13|4, 14|5, 15|6, 16|7, 25|7, 34|-7$. $\mathfrak{g}_{7,1.1(iii)}$: $12|3, 13|4, 14|5, 15|6, 23|5, 24|6, 25|-7, 34|7$. $\mathfrak{g}_{7,1.1(iv)}$: $12|3, 13|4, 15|6, 16|7, 23|5, 24|6, 25|7, 34|7$. $\mathfrak{g}_{7,1.1(v)}$: $13|4, 14|5, 15|6, 16|7, 23|5, 24|6, 25|7, 34|-7$. $\mathfrak{g}_{7,1.1(vi)}$: $12|3, 13|4, 14|5, 16|7, 23|5, 25|7, 34|-7$. $\mathfrak{g}_{7,1.2(i_\lambda)}$ $(\lambda\in\mathbb{C},\ \lambda\neq -1)$ ‡ :

* $\mathfrak{g}_{7,2.46}$ in [5] has to be removed since it is isomorphic with $\mathfrak{g}_{7,2.36}$: the y_ks $(1\leqslant k\leqslant 7)$ given by the array $(x_3, x_2, x_1, x_4, -x_6, -x_5, x_7)$ satisfy the relations of $\mathfrak{g}_{7,2.46}$ within $\mathfrak{g}_{7,2.36}$.

† These comm. rel. differ from [5], yet give an isomorphic Lie algebra.

‡ The value $\lambda=-1$ has to be removed, since for that value $\mathfrak{g}_{7,1.2(i_\lambda)}$ has rank 2 and is isomorphic to $\mathfrak{g}_{7,2.37}$.

$12|4, 13|6, 14|5, 15|7, 23|\lambda\cdot 5, 24|6, 26|7, 34|(1-\lambda)\cdot 7$. $\mathfrak{g}_{7,1.2(ii)}$: $12|4, 14|5, 15|7, 16|7, 23|6, 24|6, 25|7, 34|-7$. $\mathfrak{g}_{7,1.2(iii)}$: $12|4, 13|6, 14|5, 15|7, 23|5, 24|6, 34|-7$. $\mathfrak{g}_{7,1.2(iv)}$: $12|4,\ 14|6,\ 15|-7, 16|7, 23|5, 25|7, 34|7$. $\mathfrak{g}_{7,1.3(i_\lambda)}$ $(\lambda\in\mathbb{C})$: $12|4,\ 13|5,\ 14|6,\ 16|7,\ 23|6,\ 24|\lambda\cdot 7,\ 25|7,\ 35|7$. $\mathfrak{g}_{7,1.3(ii)}$: $12|4, 13|5, 14|6, 16|7, 23|6, 24|7, 25|(\frac{1}{2})\cdot 7, 34|(-\frac{1}{2})\cdot 7$. $\mathfrak{g}_{7,1.3(iii)}$: $12|4, 13|5, 14|6, 16|7, 24|7, 35|7$. $\mathfrak{g}_{7,1.3(iv)}$: $12|4, 13|5, 14|6, 23|6, 24|7, 35|7$. $\mathfrak{g}_{7,1.3(v)}$: $12|4, 14|6, 16|7, 23|6, 24|7, 34|-7, 35|-7$. $\mathfrak{g}_{7,1.4}$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|6, 24|7$. $\mathfrak{g}_{7,1.5}$: $12|3, 13|4, 14|5, 15|6, 23|6, 25|-7, 34|7$. $\mathfrak{g}_{7,1.6}$: $12|3, 13|4, 14|5, 15|6, 16|7, 23|7$. $\mathfrak{g}_{7,1.7}$: $12|4, 13|5, 14|6, 15|7, 23|6, 24|7$. $\mathfrak{g}_{7,1.8}$ † : $12|4, 14|6, 16|7, 23|5, 24|7, 35|7$. $\mathfrak{g}_{7,1.9}$ † : $12|4, 13|6, 14|5, 15|7, 23|7, 24|6$. $\mathfrak{g}_{7,1.10}$: $12|3, 13|4, 14|6, 16|7, 23|5, 25|7$. $\mathfrak{g}_{7,1.11}$: $12|4, 14|5, 15|6, 16|7, 23|6, 24|6, 25|7, 34|-7$. $\mathfrak{g}_{7,1.12}$ † : $12|4, 14|5, 15|6, 16|7, 23|7, 24|6, 25|7$. $\mathfrak{g}_{7,1.13}$: $12|3, 13|4, 14|6, 15|7, 16|7, 23|5, 24|7$. $\mathfrak{g}_{7,1.14}$: $12|3, 13|4, 14|6, 23|5, 25|-7, 26|-7, 34|7$. $\mathfrak{g}_{7,1.15}$: $12|4, 14|5, 15|6, 16|7, 23|7, 24|7$. $\mathfrak{g}_{7,1.16}$: $12|3, 13|5, 14|6, 15|7, 16|7, 23|7$. $\mathfrak{g}_{7,1.17}$: $12|3, 13|4, 14|6, 16|7, 23|5, 25|6, 26|7, 34|-7, 35|7$. $\mathfrak{g}_{7,1.18}$: $12|4, 14|5, 15|7, 23|(6+7), 24|6$. $\mathfrak{g}_{7,1.19}$: $12|4, 13|5,\ 14|7,\ 15|6,\ 24|6, 35|7$. $\mathfrak{g}_{7,1.20}$: $12|3, 15|6,\ 16|7,\ 23|4,\ 24|6,\ 25|7,\ 34|7$. $\mathfrak{g}_{7,1.21}$: $12|4, 14|5,\ 15|6,\ 23|6,\ 24|6,\ 26|7,\ 45|-7$.

Rank 2 : 49 algebras of which 1 continuous 1-parameter series.

$\mathfrak{g}_{7,2.1(i_\lambda)}$ $(\lambda\in\mathbb{C})$: $12|4, 13|5, 14|6, 16|7, 23|6, 25|\lambda\cdot 7, 34|(\lambda-1)\cdot 7$. $\mathfrak{g}_{7,2.1(ii)}$: $12|4, 13|5, 14|6, 16|7, 25|7, 34|7$. $\mathfrak{g}_{7,2.1(iii)}$: $12|4, 14|6, 16|7, 23|6, 25|7, 34|-7$. $\mathfrak{g}_{7,2.1(iv)}$: $13|5, 14|6, 16|7, 23|6, 25|7, 34|7$. $\mathfrak{g}_{7,2.1(v)}$: $12|4, 13|5, 14|6, 23|6, 25|7, 34|7$. $\mathfrak{g}_{7,2.2}$ † : $12|5, 13|6, 14|2\cdot 7, 23|4,\ 26|7,\ 35|-7, 36|7$. $\mathfrak{g}_{7,2.3}$: $12|3, 13|4, 14|5, 15|6,\ 16|7$. $\mathfrak{g}_{7,2.4}$: $12|3, 13|4, 14|5, 15|6, 25|-7, 34|7$. $\mathfrak{g}_{7,2.5}$: $12|3, 13|4, 15|6, 16|7, 23|5, 24|6, 34|7$. $\mathfrak{g}_{7,2.6}$: $12|3, 13|4, 14|6, 23|5, 26|7, 34|-7$. $\mathfrak{g}_{7,2.7}$: $12|3, 13|4, 14|6, 16|7, 23|5$. $\mathfrak{g}_{7,2.8}$: $12|3, 13|4, 14|6, 15|7, 23|5, 24|7$. $\mathfrak{g}_{7,2.9}$: $12|3, 13|4, 14|6, 23|5, 25|7$. $\mathfrak{g}_{7,2.10}$ † : $12|4, 13|7, 14|5, 15|6, 26|7,\ 45|-7$. $\mathfrak{g}_{7,2.11}$: $12|4, 14|5, 15|7, 23|6, 24|6$. $\mathfrak{g}_{7,2.12}$: $12|4, 13|5, 14|6, 24|7, 35|7$. $\mathfrak{g}_{7,2.13}$: $12|4, 14|5, 15|6, 23|6, 26|7, 45|-7$. $\mathfrak{g}_{7,2.14}$: $12|4, 14|5, 15|6, 16|7, 23|5, 34|-6, 35|-7$. $\mathfrak{g}_{7,2.15}$: $12|4, 14|5, 15|6, 16|7, 23|6, 34|-7$. $\mathfrak{g}_{7,2.16}$: $12|4, 14|5, 15|6, 16|7, 23|7$. $\mathfrak{g}_{7,2.17}$: $12|4, 14|5, 15|7, 23|5, 24|6, 34|-7$. $\mathfrak{g}_{7,2.18}$: $12|4, 14|5, 15|7, 23|7, 24|6$. $\mathfrak{g}_{7,2.19}$: $12|4, 13|6, 14|5, 15|7, 24|6$. $\mathfrak{g}_{7,2.20}$: $12|4, 13|5, 15|6, 16|7, 35|7$. $\mathfrak{g}_{7,2.21}$: $12|4, 13|5, 14|6, 16|7, 23|7$. $\mathfrak{g}_{7,2.22}$: $13|4, 14|6, 16|7, 23|5, 34|7$. $\mathfrak{g}_{7,2.23}$: $14|6, 16|7, 23|5, 25|7, 34|7$. $\mathfrak{g}_{7,2.24}$: $12|4, 14|6, 15|-7, 16|7, 23|5, 34|7$. $\mathfrak{g}_{7,2.25}$: $12|5, 15|6, 16|7, 23|6, 34|-7, 35|-7$. $\mathfrak{g}_{7,2.26}$: $12|4, 13|5, 15|6, 25|7, 34|7, 35|6$. $\mathfrak{g}_{7,2.27}$: $12|5, 13|7, 15|6, 24|7, 25|7$. $\mathfrak{g}_{7,2.28}$: $12|5, 13|6, 16|7, 25|7, 34|7$. $\mathfrak{g}_{7,2.29}$: $12|5, 15|6, 23|6, 25|7, 34|7$. $\mathfrak{g}_{7,2.30}$: $12|5, 15|6, 16|7, 25|7, 34|7$. $\mathfrak{g}_{7,2.31}$: $12|4, 14|6, 16|7, 23|5, 25|7$. $\mathfrak{g}_{7,2.32}$: $12|4, 13|5, 14|6, 16|7, 35|7$. $\mathfrak{g}_{7,2.33}$: $12|4, 14|6, 16|7, 23|5, 35|7$. $\mathfrak{g}_{7,2.34}$: $12|4, 13|5, 14|7, 24|6, 35|7$. $\mathfrak{g}_{7,2.35}$: $12|4, 13|5, 15|6, 24|6, 25|7, 34|7$. $\mathfrak{g}_{7,2.36}$: $13|5, 14|7, 23|6, 24|-5, 36|-7$. $\mathfrak{g}_{7,2.37}$: $12|4, 13|5, 14|5, 16|7, 24|6, 25|7, 34|7$. $\mathfrak{g}_{7,2.38}$: $12|3, 13|6, 15|7, 23|7, 24|6$. $\mathfrak{g}_{7,2.39}$: $12|4, 13|5, 14|6, 15|7, 23|6$. $\mathfrak{g}_{7,2.40}$: $12|3, 13|5, 14|7, 23|6, 24|5$. $\mathfrak{g}_{7,2.41}$: $12|3, 13|5, 14|6, 16|7, 23|6, 25|7$. $\mathfrak{g}_{7,2.42}$: $12|4, 13|5, 15|6, 23|6, 24|7$. $\mathfrak{g}_{7,2.43}$: $12|4, 13|5, 14|7, 15|6, 35|7$. $\mathfrak{g}_{7,2.44}$: $12|4, 13|5, 15|7, 23|6, 24|7$. $\mathfrak{g}_{7,2.45}$ § : $12|5, 14|7, 15|6, 23|7, 24|6$.

† These comm. rel. differ from [5], yet give an isomorphic Lie algebra.

§ There is an error in [5] where one has $12|4$ instead of $12|5$. With $12|4$, one gets a Lie algebra isomorphic to $\mathfrak{g}_{7,2.38}$ by simply permuting $x_3 \leftrightarrow x_4$ and $x_6 \leftrightarrow x_7$.

Rank 3 : 25 ‡ *algebras of which 1 continuous 1-parameter series.*

$\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda \in \mathbb{C})$: $12|4, 13|5, 16|7, 23|6, 25|\lambda \cdot 7, 34|(\lambda-1)\cdot 7$. $\mathfrak{g}_{7,3.1(ii)}$: deleted ‡. $\mathfrak{g}_{7,3.1(iii)}$: $12|4, 13|5, 16|7, 25|7, 34|7$. $\mathfrak{g}_{7,3.2}$: $12|4, 13|5, 14|6, 16|7$. $\mathfrak{g}_{7,3.3}$: $12|4, 14|6, 16|7, 23|5$. $\mathfrak{g}_{7,3.4}$: $12|4, 13|5, 24|6, 35|7$. $\mathfrak{g}_{7,3.5}$: $12|4, 13|5, 24|6, 25|7, 34|7$. $\mathfrak{g}_{7,3.6}$: $12|4, 13|5, 15|7, 23|6$. $\mathfrak{g}_{7,3.7}$: $12|5, 15|6, 24|6, 34|-7$. $\mathfrak{g}_{7,3.8}$: $12|5, 13|6, 15|7, 24|7$. $\mathfrak{g}_{7,3.9}$: $12|5, 15|-7, 23|6, 24|7$. $\mathfrak{g}_{7,3.10}$: $12|5, 13|6, 24|6, 26|7, 35|7$. $\mathfrak{g}_{7,3.11}$: $12|5, 13|6, 15|7, 24|6$. $\mathfrak{g}_{7,3.12}$: $12|5, 13|6, 24|6, 34|7$. $\mathfrak{g}_{7,3.13}$: $12|5, 15|6, 25|7, 34|7$. $\mathfrak{g}_{7,3.14}$: $12|5, 13|6, 16|7, 24|7$. $\mathfrak{g}_{7,3.15}$: $12|5, 13|6, 25|7, 34|7$. $\mathfrak{g}_{7,3.16}$: $12|5, 15|7, 34|6, 36|7$. $\mathfrak{g}_{7,3.17}$: $12|5, 15|6, 16|7, 34|7$. $\mathfrak{g}_{7,3.18}$: $12|6, 16|7, 25|7, 34|7$. $\mathfrak{g}_{7,3.19}$: $12|6, 13|7, 34|6, 45|7$. $\mathfrak{g}_{7,3.20}$: $12|4, 13|5, 14|6, 15|7$. $\mathfrak{g}_{7,3.21}$: $12|4, 13|5, 14|6, 35|7$. $\mathfrak{g}_{7,3.22}$: $12|4, 13|5, 15|7, 25|6, 34|6$. $\mathfrak{g}_{7,3.23}$: $12|3, 13|5, 14|7, 23|6$. $\mathfrak{g}_{7,3.24}$: $12|5, 23|6, 24|7, 34|5$.

Rank 4 : 4 algebras.

$\mathfrak{g}_{7,4.1}$: $12|5, 13|6, 34|7$. $\mathfrak{g}_{7,4.2}$: $12|5, 13|6, 14|7$. $\mathfrak{g}_{7,4.3}$: $12|6, 35|6, 45|7$. $\mathfrak{g}_{7,4.4}$: $14|7, 25|7, 36|7$.

For the sake of completeness, we also give the relations for the 6-dimensional indecomposable complex NLAs. For the correspondence with Morozov types, see [18] sec. 4.4.

Rank 1 : 4 algebras. $\mathfrak{g}_{6,12}$: $12|4, 14|5, 15|6, 23|6, 24|6$. $\mathfrak{g}_{6,17}$: $12|3, 13|4, 14|5, 15|6, 23|6$. $\mathfrak{g}_{6,19}$: $12|3, 13|4, 14|5, 15|6, 23|5, 24|6$. $\mathfrak{g}_{6,20}$: $12|3, 13|4, 14|5, 23|5, 25|6, 34|-6$.

Rank 2 : 10 algebras.

$\mathfrak{g}_{6,5}$: $12|4, 14|5, 23|6, 24|6$. $\mathfrak{g}_{6,7}$* : $12|4, 13|5, 14|6, 23|-6$. $\mathfrak{g}_{6,8}$: $12|4, 14|5, 23|5, 24|6$. $\mathfrak{g}_{6,10}$: $12|4, 13|5, 14|6, 35|6$. $\mathfrak{g}_{6,11}$: $12|4, 14|5, 15|6, 23|6$. $\mathfrak{g}_{6,13}$: $12|4, 14|5, 15|6, 23|5, 34|-6$. $\mathfrak{g}_{6,14}$: $12|3, 13|4, 14|5, 23|6$. $\mathfrak{g}_{6,15}$: $12|3, 13|4, 15|6, 23|5, 24|6$. $\mathfrak{g}_{6,16}$: $12|3, 13|4, 14|5, 15|6$. $\mathfrak{g}_{6,18}$: $12|3, 13|4, 14|5, 25|6, 34|-6$.

Rank 3 : 6 algebras.

$\mathfrak{g}_{6,1}$: $12|5, 14|6, 23|6$. $\mathfrak{g}_{6,2}$: $12|5, 15|6, 34|6$. $\mathfrak{g}_{6,3}$: $12|4, 13|5, 23|6$. $\mathfrak{g}_{6,4}$: $12|4, 13|6, 24|5$. $\mathfrak{g}_{6,6}$: $12|4, 23|6, 24|5$. $\mathfrak{g}_{6,9}$: $12|4, 13|5, 25|6, 34|6$.

2.3. Isomorphisms for the continuous series.

With some calculations one gets: $\mathfrak{g}_{7,0.4(\lambda')} \cong \mathfrak{g}_{7,0.4(\lambda)} \Leftrightarrow \lambda' = \pm\lambda$; $\mathfrak{g}_{7,1.1(i_{\lambda'})} \cong \mathfrak{g}_{7,1.1(i_\lambda)} \Leftrightarrow \lambda' = \lambda$; $\mathfrak{g}_{7,1.2(i_{\lambda'})} \cong \mathfrak{g}_{7,1.2(i_\lambda)} \Leftrightarrow (\lambda' = \lambda \text{ or } \lambda\lambda' = 1)$; $\mathfrak{g}_{7,1.3(i_{\lambda'})} \cong \mathfrak{g}_{7,1.3(i_\lambda)} \Leftrightarrow \lambda' = \lambda$; $\mathfrak{g}_{7,2.1(i_{\lambda'})} \cong \mathfrak{g}_{7,2.1(i_\lambda)} \Leftrightarrow \lambda' = \lambda$. As to $\mathfrak{g}_{7,3.1(i_\lambda)}$, one has:

Lemma 1. *(i)* $\mathfrak{g}_{7,3.1(i_{\lambda'})} \cong \mathfrak{g}_{7,3.1(i_\lambda)}$ *if and only if* $\lambda' = s(\lambda)$ *with* s *any element of the group of transformations of* λ : $G = \{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, 1-\frac{1}{\lambda}, \frac{\lambda}{\lambda-1}\}$, *which is isomorphic to the symmetric group* $\mathcal{S}_3$.
(ii) $j(\lambda) = \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ *is a* $(\mathbb{C}\cup\infty)$*-valued invariant for* $\mathfrak{g}_{7,3.1(i_\lambda)}$.

Proof. (i) Denote $(x'_k)_{1\leqslant k\leqslant 7}$, $(x_k)_{1\leqslant k\leqslant 7}$ the respective bases (with the prescribed commutation relations) of $\mathfrak{g}_{7,3.1(i_{\lambda'})}$, $\mathfrak{g}_{7,3.1(i_\lambda)}$. Observe first that the

‡ $\mathfrak{g}_{7,3.1(ii)}$ in [5] (with comm. rel. $\mathfrak{g}_{7,3.1(ii)}$: $12|4, 13|5, 23|6, 25|7, 34|7$.) has to be removed since it is isomorphic with $\mathfrak{g}_{7,3.1(i_\lambda)}$ for $\lambda = 0$: the y_ks $(1 \leqslant k \leqslant 7)$ given by the array $(x_2, x_3, -x_1, x_6, x_4, x_5, -x_7)$ satisfy the relations of $\mathfrak{g}_{7,3.1(ii)}$ within $\mathfrak{g}_{7,3.1(i_\lambda)}$ for $\lambda = 0$.

* Here we differ slightly from the historical notation which has 23|6 and is isomorphic by changing only $x_3 \mapsto -x_3, x_5 \mapsto -x_5$.

transposition $\tau_1 = (1,2)$ (resp. $\tau_2 = (2,3)$) generates an isomorphism $(\tilde{\tau}_1)_{(\lambda',\lambda)}$ (resp. $(\tilde{\tau}_2)_{(\lambda',\lambda)}$): $\mathfrak{g}_{7,3.1(i_{\lambda'})} \to \mathfrak{g}_{7,3.1(i_\lambda)}$ such that $(\tilde{\tau}_1)_{(\lambda',\lambda)}(x'_i) = x_{\tau_1(i)}$ (resp. $(\tilde{\tau}_2)_{(\lambda',\lambda)}(x'_i) = x_{\tau_2(i)}$) $\forall i\, 1 \leqslant i \leqslant 3$ if and only if $\lambda' = s_1(\lambda) = \frac{1}{\lambda}$ (resp. $\lambda' = s_2(\lambda) = 1-\lambda$). Now τ_1, τ_2 generate $\mathcal{S}_3$ and s_1, s_2 generate $G \cong \mathcal{S}_3$. It follows that there exists an isomorphism $f : \mathfrak{g}_{7,3.1(i_{\lambda'})} \to \mathfrak{g}_{7,3.1(i_\lambda)}$ such that $f(x'_i) = x_{\sigma(i)}\, \forall i\, 1 \leqslant i \leqslant 3$ for some fixed $\sigma \in \mathcal{S}_3$ if and only if $\lambda' = s(\lambda)$ for some $s \in G$. The same holds true with $f(x'_i) = \alpha_i x_{\sigma(i)}\, \forall i\, 1 \leqslant i \leqslant 3$ $\alpha_i \neq 0$, since for all $u_1, u_2, u_3 \neq 0$, $\mathrm{diag}(u_1, u_2, u_3, u_1u_2, u_1u_3, u_2u_3, u_1u_2u_3)$ $\in \mathrm{Aut}(\mathfrak{g}_{7,3.1(i_\lambda)})$. In particular, the cases $\lambda = 0, \lambda = 1$ are isomorphic, and so are the 3 cases $\lambda = -1, \lambda = 2, \lambda = \frac{1}{2}$. Now, note first that for any λ the derivations $\mathrm{diag}(\xi_1^1, \xi_2^2, \xi_3^3, \xi_1^1 + \xi_2^2, \xi_1^1 + \xi_3^3, \xi_2^2 + \xi_3^3, \xi_1^1 + \xi_2^2 + \xi_3^3)$ form a diagonalized maximal torus T_λ of $\mathfrak{g}_{7,3.1(i_\lambda)}$. The associated weight system is $(\beta_1, \beta_2, \beta_3, \beta_1+\beta_2, \beta_1+\beta_3, \beta_2+\beta_3, \beta_1+\beta_2+\beta_3)$. Then suppose that there exists an isomorphism $f : \mathfrak{g}_{7,3.1(i_{\lambda'})} \to \mathfrak{g}_{7,3.1(i_\lambda)}$ and denote $y_i = f(x'_i)\, \forall i\, 1 \leqslant i \leqslant 3$. $f \circ T_{\lambda'} \circ f^{-1}$ is a maximal torus of $\mathfrak{g}_{7,3.1(i_\lambda)}$. By Mostow's theorem [20], there exists $\varphi \in Aut\, \mathfrak{g}_{7,3.1(i_\lambda)}$ such that $f \circ T_{\lambda'} \circ f^{-1} = \varphi^{-1} \circ T_\lambda \circ \varphi$. Hence replacing f by $\varphi \circ f$, one may suppose $f \circ T_{\lambda'} \circ f^{-1} = T_\lambda$, i.e. f intertwines $T_{\lambda'}$ and T_λ. Now, as the fundamental weights $\beta_1, \beta_2, \beta_3$ are simple, there exist nonzero constants α_i and $\sigma \in \mathcal{S}_3$ such that $y_i = \alpha_i x_{\sigma(i)}\, \forall i\, 1 \leqslant i \leqslant 3$ (take $t' \in T_{\lambda'}$ with simple eigenvalues, e.g. $\mathrm{diag}(1,2,4,3,5,6,7)$; since x'_i is an eigenvector of t', y_i is an eigenvector of $f \circ t' \circ f^{-1}$ which is also diagonal with the same simple eigenvalues; as $y_i \notin [\mathfrak{g}_{7,3.1(i_\lambda)}, \mathfrak{g}_{7,3.1(i_\lambda)}]$, necessarily y_i is a nonzero scalar multiple of one of x_1, x_2, x_3) whence $\lambda' = s(\lambda)$ for some $s \in G$. The result follows.
(ii) Follows from the fact that ([21], p.50 ex. 3) the field of invariants of rational functions of λ under the action of G is generated by j.

Remark. The invariant given in [27] for 147_E which corresponds to $\lambda \notin \{0,1\} \cup \{-1, 2, \frac{1}{2}\}$ is simply $K(\lambda) = -\frac{j(\lambda)}{j(2)-j(\lambda)}$. As noted in [27] and in [21] p. 32, there seems to exist interesting connections with the theory of binary quartics. It should also be noted that the 3 sets of values $\{0,1\}$, $\{-1,2,\frac{1}{2}\}$, $\{-\omega, -\omega^2\}$ ($\omega = \exp\frac{2i\pi}{3}$) are precisely the values where gaps occur both in adjoint and trivial cohomology.

2.4. Correspondence between Carles' and Seeley's notations.

There are 129 indecomposable NLAs in Seeley's classification (after removal of 13457_H which is not a Lie algebra). Now 8 of them (247_P,2357_B,147_C,1357_K,1357_P,1357_Q,12457_M,123457_G) are members of Carles' continuous series considered separately, and 2 of them (1357_S, 12457_N) each correspond to 2 Carles' NLAs. Hence we get $129 - 8 + 2 = 123$ NLAs as in Carles' list. See [27] for the commutation relations of the basis elements a, b, c, d, e, f, g of the algebras in Seeley's classification. We denote " = " when the commutation relations are identical; " $\cong$ " when there is a (nonidentity) isomorphism $f : \mathfrak{g}_{7,*.*} \to **_*$, and then we give f as an array

$(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$ with $y_k = f(x_k)$, that is the y_ks in the array satisfy Carles' relations within Seeley's algebra. $i = \sqrt{-1}$.

$37_A \cong \mathfrak{g}_{7,4.2} : (b, a, c, d, -e, f, g)$. $37_B \cong \mathfrak{g}_{7,4.1} : (b, a, c, d, -e, f, g)$. $37_C = \mathfrak{g}_{7,3.24}$. $37_D \cong \mathfrak{g}_{7,3.12} : (a, c, b, d, f, e, g)$.

$357_A \cong \mathfrak{g}_{7,3.6} : (a, d, b, g, c, -f, e)$. $357_B = \mathfrak{g}_{7,3.23}$. $357_C = \mathfrak{g}_{7,2.40}$.

$27_A \cong \mathfrak{g}_{7,4.3} : (c, d, a, b, e, f, g)$. $27_B \cong \mathfrak{g}_{7,3.19} : (d, a, a + b - c, e, -b, -f, g)$.

$257_A \cong \mathfrak{g}_{7,3.8} : (a, b, e, d, c, g, f)$. $257_B \cong \mathfrak{g}_{7,3.11} : (a, b, e, d, c, g, f)$. $257_C \cong \mathfrak{g}_{7,3.9} : (a, b, e, -d, c, g, -f)$. $257_D \cong \mathfrak{g}_{7,2.45} : (a, b, e, d, c, f, g)$. $257_E \cong \mathfrak{g}_{7,3.15} : (b, a, d, -e, -c, g, -f)$. $257_F \cong \mathfrak{g}_{7,3.14} : (a, d, b, e, g, c, f)$. $257_G \cong \mathfrak{g}_{7,2.36} : (-e, ib, a, id, g, -ic, if)$. $257_H \cong \mathfrak{g}_{7,3.7} : (a, b, e, d, c, f, g)$. $257_I \cong \mathfrak{g}_{7,2.27} : (a, b, e, c - d, c, f, g)$. $257_J = \mathfrak{g}_{7,2.38}$. $257_K \cong \mathfrak{g}_{7,3.13} : (b, a, e, d, -c, -g, -f)$. $257_L \cong \mathfrak{g}_{7,2.29} : (-b, a, -d, -e, c, -g, f)$.

$247_A = \mathfrak{g}_{7,3.20}$. $247_B = \mathfrak{g}_{7,3.21}$. $247_C = \mathfrak{g}_{7,2.43}$. $247_D = \mathfrak{g}_{7,3.22}$. $247_E \cong \mathfrak{g}_{7,2.12} : (a, b + c, i(-b + c), d + e, i(-d + e), 2g, 2f)$. $247_F = \mathfrak{g}_{7,3.4}$. $247_G = \mathfrak{g}_{7,2.34}$. $247_H = \mathfrak{g}_{7,1.19}$. $247_I \cong \mathfrak{g}_{7,3.5} : (a, c, b, e, d, g, f)$. $247_J \cong \mathfrak{g}_{7,2.26} : (a - c, b, -c, d, -e, g, -f)$. $247_K \cong \mathfrak{g}_{7,2.35} : (a, c, b, e, d, g, f)$. $247_L = \mathfrak{g}_{7,2.39}$. $247_M = \mathfrak{g}_{7,2.42}$. $247_N = \mathfrak{g}_{7,2.44}$. $247_O = \mathfrak{g}_{7,1.7}$. $247_P \cong \mathfrak{g}_{7,3.1(i_\lambda)}, \lambda = 0 : (-c, a, b, e, f, d, -g)$. $247_Q \cong \mathfrak{g}_{7,2.1(v)} : (a, c, -b, e, -d, f, -g)$. $247_R \cong \mathfrak{g}_{7,1.3(iv)} : (a, -i(b + c), -b + c, -i(d + e), -d + e, -2if, -2g)$.

$2457_A \cong \mathfrak{g}_{7,3.2} : (a, b, e, c, g, d, f)$. $2457_B \cong \mathfrak{g}_{7,3.3} : (a, b, e, c, f, d, g)$. $2457_C \cong \mathfrak{g}_{7,2.21} : (a, b, e, c, g, d, f)$. $2457_D \cong \mathfrak{g}_{7,1.16} : (a, b, c, c - e, d, d - g, f)$. $2457_E \cong \mathfrak{g}_{7,2.11} : (a, b, e, c, d, f, g)$. $2457_F \cong \mathfrak{g}_{7,2.20} : (a, e, b, g, c, d, f)$. $2457_G \cong \mathfrak{g}_{7,2.19} : (a, b, e, c, d, f, g)$. $2457_H \cong \mathfrak{g}_{7,2.18} : (a, b, e, c, d, f, g)$. $2457_I \cong \mathfrak{g}_{7,2.22} : (a, e, b, c, -g, d, f)$. $2457_J \cong \mathfrak{g}_{7,1.18} : (a, -b, e, -c, -d, f, -f - g)$. $2457_K \cong \mathfrak{g}_{7,1.9} : (a, b, e, c, d, f, g)$. $2457_L = \mathfrak{g}_{7,2.9}$. $2457_M \cong \mathfrak{g}_{7,2.8} : (a, b, c, d, e, g, f)$.

$2357_A \cong \mathfrak{g}_{7,2.24} : (a, b, -d, c, -e - g, e, f)$. $2357_B \cong \mathfrak{g}_{7,2.1(i_\lambda)}, \lambda = 0 : (a, b, d, c, g, e, f)$. $2357_C \cong \mathfrak{g}_{7,2.17} : (a, b, d, c, e, g, f)$. $2357_D \cong \mathfrak{g}_{7,1.2(iii)} : (2a + b, b, -2c + 4d, 2c, 4e + 2g, 2g, 8f)$.

$23457_A \cong \mathfrak{g}_{7,2.7} : (a, b, c, d, g, e, f)$. $23457_B \cong \mathfrak{g}_{7,2.6} : (a, b, c, d, g, e, f)$. $23457_C \cong \mathfrak{g}_{7,2.4} : (a, b, c, d, e, f, -g)$. $23457_D \cong \mathfrak{g}_{7,1.5} : (a, b, c, d, e, f, -g)$. $23457_E \cong \mathfrak{g}_{7,1.13} : (a, b, c, d, e + g, e, f)$. $23457_F \cong \mathfrak{g}_{7,1.14} : (a, b, c, d, e + g, e, -f)$. $23457_G \cong \mathfrak{g}_{7,1.1(iii)} : (a, b, c, d, e, f, -g)$.

$17 \cong \mathfrak{g}_{7,4.4} : (a, c, e, b, d, f, g)$.

$157 \cong \mathfrak{g}_{7,3.18} : (a, b, e, f, d, c, g)$.

$147_A = \mathfrak{g}_{7,3.1(iii)}$. $147_B \cong \mathfrak{g}_{7,2.28} : (a, c, b, f, e, d, g)$. $147_C \cong \mathfrak{g}_{7,3.1(i_\lambda)}, \lambda = -1 : (c, b, a, -e, f, -d, -\frac{1}{2}g)$. $147_D \cong \mathfrak{g}_{7,2.2} : (a, 2b, c, 2e, 2d, -f, -g)$. $147_E(\xi)(\xi \notin \{0, 1\}^* \cup \{-1, 2, \frac{1}{2}\}) \cong \mathfrak{g}_{7,3.1(i_\lambda)}, \lambda = \xi : (a, b, -c, d, f, -e, g)$.

$1457_A \cong \mathfrak{g}_{7,3.17} : (a, b, e, f, c, d, g)$. $1457_B \cong \mathfrak{g}_{7,2.30} : (a, b, e, f, c, d, g)$.

$137_A = \mathfrak{g}_{7,3.16}$. $137_B{}^\dagger \cong \mathfrak{g}_{7,2.23} : (a, c, -d, -b, -f, -e, -g)$. $137_C \cong \mathfrak{g}_{7,3.10} : (c, a, d, b, -f, e, g)$. $137_D \cong \mathfrak{g}_{7,2.1(iv)} : (a, d, -c, b, -f, e, g)$.

$1357_A \cong \mathfrak{g}_{7,2.1(iii)} : (a, b, d, c, f, e, g)$. $1357_B \cong \mathfrak{g}_{7,2.25} : (a, b, d, -f, c, e, g)$. $1357_C \cong \mathfrak{g}_{7,1.3(v)} : (a, b, d, c, -f, e, g)$. $1357_D \cong \mathfrak{g}_{7,2.1(ii)} : (a, b, d, c, f, e, g)$. $1357_E \cong \mathfrak{g}_{7,2.32} : (a, b, d, c, f, e, g)$. $1357_F \cong \mathfrak{g}_{7,1.3(iii)} :$

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* As in [8] $\xi = 0, 1$ have to be excluded as giving an algebra isomorphic to 247_P.

† As observed in [8] the correct relations for 137_B are $[a, b] = e, [a, e] = g, [b, d] = g$ (instead of $[b, c] = g$), $[c, d] = f, [c, f] = g$.

(a,b,d,c,f,e,g). $1357_G \cong \mathfrak{g}_{7,2.31} : (a,b,d,c,f,e,g)$. $1357_H \cong \mathfrak{g}_{7,1.2(iv)} : (a,-b,-d,-c,f,-e,-g)$. $1357_I \cong \mathfrak{g}_{7,2.33} : (a,b,d,c,f,e,g)$. $1357_J \cong \mathfrak{g}_{7,1.8} : (a,b,d,c,f,e,g)$. $1357_K \cong \mathfrak{g}_{7,2.1(i_\lambda)}, \lambda = \frac{1}{2} : (a,b,d,c,f,e,g)$. $1357_L \cong \mathfrak{g}_{7,1.3(ii)} : (a,b,d,c,f,e,g)$. $1357_M(\xi)(\xi \neq 0, \frac{1}{2}) \cong \mathfrak{g}_{7,2.1(i_\lambda)}, \lambda = \xi : (a,b,d,c,f,e,g)$. $1357_N(\xi) \cong \mathfrak{g}_{7,1.3(i_\lambda)}, \lambda = \xi : (a, b+d, d, c+f, f, e, g)$. $1357_O = \mathfrak{g}_{7,2.41}$. $1357_P \cong \mathfrak{g}_{7,1.2(i_\lambda)}, \lambda = 0 : (a,b,d,c,e,f,g)$. $1357_Q \cong \mathfrak{g}_{7,1.2(i_\lambda)}, \lambda = 1 : (a+b, -a+b, 2(c-d), 2(c+d), 2(e+f), 2(-e+f), 4g)$. $1357_R \cong \mathfrak{g}_{7,2.37} : (b, a, -c, -c-d, -f, -e, -g)$. $1357_S(\xi)(\xi \neq 1)$[‡] : Case 1 $(\xi \neq 0)$ $1357_S(\xi) \cong \mathfrak{g}_{7,1.2(i_\lambda)}, \lambda = \frac{v+1}{v-1}$, v any square root of ξ [§]: $(va+b, u(-va+b), 2\frac{\xi+v}{v-1}(c-d), 2uv(c+d), 2uv(ve+f), 2\frac{\xi+v}{v-1}(-ve+f), 4(v+1)\xi ug)$ with u any square root of λ. Case 2 $(\xi = 0)$ $1357_S(\xi) \cong \mathfrak{g}_{7,1.2(ii)} : (a,b,c,c+d,e,f,g)$.

$13457_A \cong \mathfrak{g}_{7,2.16} : (a,b,f,c,d,e,g)$. $13457_B \cong \mathfrak{g}_{7,1.15} : (a,b,f,c,d,e,g)$. $13457_C \cong \mathfrak{g}_{7,2.10} : (a,b,f,c,d,e,g)$. $13457_D \cong \mathfrak{g}_{7,1.12} : (a,b,f,c,d,e,g)$. $13457_E = \mathfrak{g}_{7,1.1(vi)}$. $13457_F \cong \mathfrak{g}_{7,1.10} : (a,b,c,d,f,e,g)$. $13457_G = \mathfrak{g}_{7,1.03}$. 13457_H : deleted [¶]. $13457_I \cong \mathfrak{g}_{7,0.7} : (b,a,-c,-f,-d,-e,-g)$.

$12457_A \cong \mathfrak{g}_{7,2.15} : (a,b,e,c,d,f,g)$. $12457_B \cong \mathfrak{g}_{7,1.01(ii)} : (a,b,e,c,d,f,g)$. $12457_C \cong \mathfrak{g}_{7,2.13} : (a,b,e,c,d,f,g)$. $12457_D \cong \mathfrak{g}_{7,1.20} : (b,a,-c,-d,-e,-f,-g)$. $12457_E \cong \mathfrak{g}_{7,1.11} : (a,b,e,c,d,f,g)$. $12457_F \cong \mathfrak{g}_{7,1.21} : (a,b,e,c,d,f,g)$. $12457_G \cong \mathfrak{g}_{7,0.8} : (a,b,e,c,d,f,g)$. $12457_H = \mathfrak{g}_{7,2.5}$. $12457_I = \mathfrak{g}_{7,1.1(iv)}$. $12457_J = \mathfrak{g}_{7,0.6}$. $12457_K \cong \mathfrak{g}_{7,1.02} : (b,a,-c,-e,-d,-f,-g)$. $12457_L \cong \mathfrak{g}_{7,1.17} : (a+ib, ia+b, 2c, 2(d+ie), 2(id+e), 4if, 4(i-1)g)$. $12457_M \cong \mathfrak{g}_{7,0.4(\lambda)}, \lambda = -2i : (\frac{i}{4}(a+b), \frac{1}{4}(-a+b), \frac{i}{8}c, -\frac{1}{32}(d+e), \frac{i}{32}(-d+e), \frac{i}{128}(-2f+g), \frac{1}{128}g)$. $12457_N(\xi)(\xi \neq 0)$: Case 1 $(\xi \neq 1)$ $12457_N(\xi) \cong \mathfrak{g}_{7,0.4(\lambda)}, \lambda = 2i\frac{\xi+1}{\xi-1}$ [‖] : $(-i\frac{\xi-1}{4}(a+b), \frac{\xi-1}{4}(a-b), i\frac{(\xi-1)^2}{8}c, \frac{(\xi-1)^3}{32}(d+e), i\frac{(\xi-1)^3}{32}(d-e), i\frac{(\xi-1)^4}{128}(-2f+(\xi+1)g)), -\frac{(\xi-1)^5}{128}g)$. Case 2 $(\xi = 1)$ $12457_N(\xi) \cong \mathfrak{g}_{7,0.5} : (a+b, i(a-b), -2ic, -2i(d+e), 2(d-e), -4i(f-g), -8ig)$.

$12357_A \cong \mathfrak{g}_{7,2.14} : (a,b,d,c,e,f,g)$. $12357_B \cong \mathfrak{g}_{7,1.01(i)} : (a,b,d,c,e,f,g)$. $12357_C \cong \mathfrak{g}_{7,1.1(v)} : (a,-d,-b,-c,-e,-f,-g)$.

$123457_A = \mathfrak{g}_{7,2.3}$. $123457_B = \mathfrak{g}_{7,1.6}$. $123457_C = \mathfrak{g}_{7,1.1(ii)}$. $123457_D = \mathfrak{g}_{7,1.4}$. $123457_E = \mathfrak{g}_{7,0.3}$. $123457_F = \mathfrak{g}_{7,0.1}$. $123457_G = \mathfrak{g}_{7,1.1(i_\lambda)}$, $\lambda = 1$. $123457_H = \mathfrak{g}_{7,0.2}$. $123457_I(\xi)(\xi \neq 1) = \mathfrak{g}_{7,1.1(i_\lambda)}$, $\lambda = \xi$.

(Finally, see [10] for the correspondance between the notations of Goze-Khakimdjanov and Seeley.)

2.5. Cohomology.

Adjoint and trivial cohomologies for all complex 7-dimensional indecomposable NLAs, along with their weight systems under the action of the maximal torus of $\mathfrak{g}$ have been computed in [18] (see also [19], [15], and for trivial

‡ As in [8] the value $\xi = 1$ has to be excluded as it gives an algebra isomorphic to 2357_D.

§ There are 3 missing values for $\lambda : \lambda = 1$ $(v = \infty)$ which refer to 1357_Q, $\lambda = 0$ $(v = -1)$ which refer to 1357_P and $\lambda = -1$ $(v = 0)$ which is the value dismissed for $\mathfrak{g}_{7,1.2(i_\lambda)}$.

¶ As in [8] this is not a Lie algebra as Jacobi identity is not satisfied for the triple (a,b,c).

‖ Note that the value $\lambda = 2i$ is missing. This is in line with the fact that $\mathfrak{g}_{7,0.4(\lambda')} \cong \mathfrak{g}_{7,0.4(\lambda)}$ if and only if $\lambda' = \pm\lambda$ and that for $\lambda = -2i$ we got 12457_M.

cohomology [3]). It is a fact that, for the Lie algebras under consideration, the trivial cohomology is ineffective in separating nonisomorphic Lie algebras and doesn't refine the classification by weight systems of the algebras. However, the adjoint cohomology does separate all but 11 pairs of algebras, and refines the classification by weight systems of the algebras, with only 4 exceptions. From now on, we speak only of adjoint cohomology.

For any fixed continuous series, the cohomology is the same for all but some *singular values* of the parameter at which gaps occur. Here are the singular values for the 6 continuous series ($\omega = \exp \frac{2i\pi}{3}$):

algebra	singular values
$\mathfrak{g}_{7,0.4(\lambda)}$:	no singular value
$\mathfrak{g}_{7,1.1(i_\lambda)}$:	$\lambda = 1, \lambda = 3, \lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,1.2(i_\lambda)}(\lambda \neq -1)$	$\lambda = 1, \lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,1.3(i_\lambda)}$	$\lambda = 1;$
$\mathfrak{g}_{7,2.1(i_\lambda)}$	$\lambda = 0, \lambda = \frac{1}{2}, \lambda = 2, \lambda \in \{-\omega, -\omega^2\}$
$\mathfrak{g}_{7,3.1(i_\lambda)}$	$\lambda = 0, 1,\ \lambda = -1, 2, \frac{1}{2},\ \lambda \in \{-\omega, -\omega^2\}$

For continuous series, the term *generic* will refer to the values of the parameter which are not singular. The non isomorphic NLAs having the same cohomology come in 11 pairs, as shown in the following table:

#	pair	cohomology
1	$(\mathfrak{g}_{7,0.4(\lambda)},\ \mathfrak{g}_{7,0.5})$	$(1, 4, 9, 15, 16, 12, 7, 2)$
2	$(\mathfrak{g}_{7,1.1(i_\lambda)}\ (\lambda = -\omega),\ \mathfrak{g}_{7,1.1(i_\lambda)}\ (\lambda = -\omega^2))$	$(1, 4, 9, 14, 16, 12, 6, 2)$
3	$(\mathfrak{g}_{7,1.2(i_\lambda)}(\text{generic}),\ \mathfrak{g}_{7,1.2(ii)})$	$(1, 6, 16, 27, 30, 23, 12, 3)$
4	$(\mathfrak{g}_{7,1.3(i_\lambda)}(\text{generic}),\ \mathfrak{g}_{7,1.3(iii)})$	$(1, 7, 19, 30, 32, 23, 11, 3)$
5	$(\mathfrak{g}_{7,1.20},\ \mathfrak{g}_{7,1.21})$	$(1, 5, 12, 17, 17, 15, 10, 3)$
6	$(\mathfrak{g}_{7,2.1(i_\lambda)}(\text{generic}),\ \mathfrak{g}_{7,1.3(ii)})$	$(1, 8, 20, 31, 33, 24, 12, 3)$
7	$(\mathfrak{g}_{7,2.1(i_\lambda)}(\lambda = 0),\ \mathfrak{g}_{7,2.11})$	$(2, 9, 21, 33, 34, 24, 12, 3)$
8	$(\mathfrak{g}_{7,2.1(i_\lambda)}(\lambda = -\omega),\ \mathfrak{g}_{7,2.1(i_\lambda)}(\lambda = -\omega^2))$	$(1, 8, 20, 31, 34, 25, 12, 3)$
9	$(\mathfrak{g}_{7,2.26},\ \mathfrak{g}_{7,3.4})$	$(2, 8, 19, 32, 36, 26, 12, 3)$
10	$(\mathfrak{g}_{7,2.31},\ \mathfrak{g}_{7,2.41})$	$(1, 7, 18, 29, 32, 24, 12, 3)$
11	$(\mathfrak{g}_{7,2.34},\ \mathfrak{g}_{7,2.35})$	$(2, 7, 18, 29, 33, 25, 11, 3)$

As we see, adjoint cohomology refines the classification by weight systems of the NLAs, except for the 4 pairs # 2,3,4,8. In either of those 4 pairs, the weight system on the cohomology is identical for the 2 components [18].

2.6. The method of adjoining a derivation.

2.6.1.
For any n-dimensional complex NLA $\mathfrak{g}$ and any nilpotent derivation $\delta \in \mathrm{Der}\,(\mathfrak{g})$ we denote by $\tilde{\mathfrak{g}}_\delta$ the NLA we get by adjoining the derivation δ to $\mathfrak{g}$, that is the semi-direct product $\tilde{\mathfrak{g}}_\delta = \mathbb{C}\delta \oplus \mathfrak{g}$. We say that δ, δ' are projectively equivalent (notation $\delta' \sim \delta$) if and only if there exist $\varphi \in \mathrm{Aut}\,\mathfrak{g}$ and $k \in \mathbb{C}\backslash\{0\}$ such that $\delta' = k\,\varphi^{-1}\,\delta\,\varphi \mod \mathrm{ad}(\mathfrak{g})$. From the following lemma, which can already be found in [4] (see also [14]), this simply means that there exists an isomorphism $\psi : \tilde{\mathfrak{g}}_{\delta'} \to \tilde{\mathfrak{g}}_\delta$ leaving $\mathfrak{g}$ invariant.

Lemma 2. *Let $\mathfrak{g}$ be a NLA, δ,δ' nilpotent derivations of $\mathfrak{g}$. There exists an isomorphism $\psi : \tilde{\mathfrak{g}}_{\delta'} \to \tilde{\mathfrak{g}}_\delta$ such that $\psi(\mathfrak{g}) = \mathfrak{g}$ if and only if $\delta' \sim \delta$.*

Proof. Let $\psi : \tilde{\mathfrak{g}}_{\delta'} \to \tilde{\mathfrak{g}}_\delta$ an isomorphism such that $\psi(\mathfrak{g}) = \mathfrak{g}$. Denote $\varphi = \psi|_{\mathfrak{g}} \in \mathrm{Aut}\,\mathfrak{g}$. As ψ is bijective, there exist $k \in \mathbb{C} \setminus \{0\}$ and $a \in \mathfrak{g}$ such that $\psi(\delta') = k\delta + a$. Then, for any $x \in \mathfrak{g}$, on one hand $[\psi(\delta'), x] = \psi[\delta', \varphi^{-1}(x)] = \psi(\delta'(\varphi^{-1}(x))) = \varphi(\delta'(\varphi^{-1}(x))) = (\varphi\delta'\varphi^{-1})(x)$, and on the other hand $[\psi(\delta'), x] = [k\delta + a, x] = k\delta(x) + (\mathrm{ad}\ a)(x)$, whence $\varphi\delta'\varphi^{-1} = k\delta + \mathrm{ad}\ a$ i.e. $\delta' = k\varphi^{-1}\delta\varphi + \mathrm{ad}\ \varphi^{-1}(a)$. Hence $\delta' \sim \delta$.
Conversely, let δ,δ' nilpotent derivations of $\mathfrak{g}$ with $\delta' \sim \delta$. There exist $\varphi \in \mathrm{Aut}\,\mathfrak{g}$, $k \in \mathbb{C}\backslash\{0\}$ and $a \in \mathfrak{g}$ such that $\delta' = k\,\varphi^{-1}\,\delta\,\varphi + \mathrm{ad}\ a$. Define $\psi : \tilde{\mathfrak{g}}_{\delta'} \to \tilde{\mathfrak{g}}_\delta$ by $\psi(\delta') = k\delta + \varphi(a)$ and $\psi|_{\mathfrak{g}} = \varphi$. To prove that ψ is an isomorphism, one has only to prove that $\psi[\delta', \varphi^{-1}(x)] = [\psi(\delta'), x] \forall x \in \mathfrak{g}$. Now, for any $x \in \mathfrak{g}$, $\psi[\delta', \varphi^{-1}(x)] = \psi(\delta'(\varphi^{-1}(x))) = \psi\left(k\varphi^{-1}(\delta(x)) + [a, \varphi^{-1}(x)]\right) = k\delta(x) + [\varphi(a), x] = [k\delta + \varphi(a), x] = [\psi(\delta'), x]$ hence ψ is an isomorphism indeed.

One may also consider $\tilde{\mathfrak{g}}_\delta$ as the extension

$$0 \longrightarrow \mathfrak{g} \xrightarrow{\ i\ } \tilde{\mathfrak{g}}_\delta \xrightarrow{\ p\ } \mathbb{C} \longrightarrow 0 \tag{1}$$

with $i(x) = (0, x)$, $p((\lambda\delta, x)) = \lambda$, $\forall \lambda \in \mathbb{C}, x \in \mathfrak{g}$. Then $\delta' \sim \delta$ if and only if the extensions are quasi-equivalent, that is if there exist isomorphisms φ, ψ, L such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r, "i"] \arrow[d, "\varphi"'] & \tilde{\mathfrak{g}}_\delta \arrow[r, "p"] \arrow[d, "\psi"'] & \mathbb{C} \arrow[r] \arrow[d, "L"'] & 0 \arrow[d] \\
0 \arrow[r] & \mathfrak{g} \arrow[r, "i'"] & \tilde{\mathfrak{g}}_{\delta'} \arrow[r, "p'"] & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

Now, the action $(\varphi, \delta) \mapsto \varphi^{-1}\delta\varphi$ of $\mathrm{Aut}\,\mathfrak{g}$ on $\mathrm{Der}\,(\mathfrak{g})$ passes to the the adjoint cohomology group $H^1(\mathfrak{g})$ and the projective adjoint cohomology group $PH^1(\mathfrak{g})$. Then for any nilpotent δ, δ', $\delta' \sim \delta$ simply means that the canonical images of δ, δ' in $PH^1(\mathfrak{g})$ are on the same $\mathrm{Aut}\,\mathfrak{g}$-orbit. In that way the nilpotent orbits in $PH^1(\mathfrak{g})$ classify the quasi-equivalent extensions (1).

2.6.2.

If we vary δ all over nilpotent derivations of $\mathfrak{g}$, the $\tilde{\mathfrak{g}}_\delta$s will give all $(n+1)$-dimensional complex NLAs containing $\mathfrak{g}$. We call the set of all $(n+1)$-dimensional complex indecomposable NLAs containing $\mathfrak{g}$ the *strain* of $\mathfrak{g}$. To compute (up to isomorphism) the members of the strain of $\mathfrak{g}$, the following steps have to be completed ($(n+1)$-dimensional direct products are dismissed).
Step 1) Determine the most general nilpotent derivation $\delta \mod \mathrm{ad}(\mathfrak{g})$ of $\mathfrak{g}$ in terms of complex parameters (we call it "generic nilpotent derivation mod $\mathrm{ad}(\mathfrak{g})$").
Step 2) Compute $\mathrm{Aut}\,\mathfrak{g}$.
Step 3) Use iteratively projective equivalence to reduce the number of parameters in δ. This entails separating several subcases, thus getting a set $\mathfrak{D}_{\mathrm{red}}$ of derivations still depending possibly on some parameters. Some cases split naturally as non proj. equiv. (e.g. if there are 2 kinds of automorphisms and the action of any first (resp. 2d) kind automorphism φ changes the parameters ξ_3^1, ξ_2^3 of δ to $b_1^1\xi_3^1, b_2^2\xi_2^3$, (resp. $b_2^1\xi_2^3, b_1^2\xi_3^1$) with $b_1^1 b_2^2 \neq 0$ (resp. $b_2^1 b_1^2 \neq 0$) from $\det\varphi$, we get a 3 cases-split). In other cases, there is no natural split. Then, to actually start the reduction, one has to externally suppose that some parameters of the derivation are zero or not, and in some instances, impose zero value to a few entries of the matrix of the automorphism (e.g. $\mathfrak{g}_{6,1}$ case2N4, $\mathfrak{g}_{6,3}$ case3N1). As a consequence, it can happen that certain derivations in $\mathfrak{D}_{\mathrm{red}}$ still are projectively equivalent.
Step 4) For each $\delta_1 \in \mathfrak{D}_{\mathrm{red}}$, match $\tilde{\mathfrak{g}}_{\delta_1}$ to some $(n+1)$-dimensional Lie algebra if a list of $(n+1)$-dimensional NLAs is available.
Step 5) If $\delta_1, \delta_2 \in \mathfrak{D}_{\mathrm{red}}$ give rise to isomorphic $\tilde{\mathfrak{g}}_{\delta_1} \cong \tilde{\mathfrak{g}}_{\delta_2}$, check whether or not they are projectively equivalent.

2.6.3.

In this paper, we apply the preceding scheme in dimension 6, thus getting the strain of each of the 30 6-dimensional complex NLAs. A recap of the results is presented as a cross table. We also present in detail 4 individual cases of interest. For a full treatment of the details of all of the other 26 cases, see the companion paper [16].

For step 1), all derivations of 6-dimensional NLAs have been computed in [18]. Subtracting adjoint derivations, we kill some parameters and get the generic nilpotent derivation mod $\mathrm{ad}(\mathfrak{g})$. For example, in the case of $\mathfrak{g}_{6,1}$ the general nilpotent derivation is

$$\delta = \begin{pmatrix} \xi_1^1 & \xi_2^1 & 0 & 0 & 0 & 0 \\ \xi_1^2 & -\xi_1^1 & 0 & 0 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & \xi_1^1 & -\xi_2^1 & 0 & 0 \\ \xi_1^4 & \xi_2^4 & -\xi_1^2 & -\xi_1^1 & 0 & 0 \\ \xi_1^5 & \xi_2^5 & \xi_3^5 & \xi_4^5 & 0 & 0 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & \xi_2^4 - \xi_1^3 & 0 \end{pmatrix} \text{ with } A = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & -\xi_1^1 \end{pmatrix} \text{ nilpotent,}$$

and since $\mathrm{ad}(x_1) = \delta_2^5 + \delta_4^6, \mathrm{ad}(x_2) = -\delta_1^5 + \delta_3^6, \mathrm{ad}(x_3) = -\delta_2^6, \mathrm{ad}(x_4) = -\delta_1^6$ (where δ_j^i is the derivation with $\xi_j^i = 1$, others 0), we kill the "adjoint" parameters $\xi_1^5, \xi_2^5, \xi_1^6, \xi_2^6$ by subtracting $\xi_2^5 \operatorname{ad}(x_1) - \xi_1^5 \operatorname{ad}(x_2) - \xi_2^6 \operatorname{ad}(x_3) - \xi_1^6 \operatorname{ad}(x_4)$.

Step 2), is performed either by hand or by the specific programs `calculautom**.red`. These programs solve the equations $i, j|k$ ($1 \leqslant i, j, k \leqslant 6$) obtained by projecting on x_k the equation $[\varphi(x_i), \varphi(x_j)] - \varphi([x_i, x_j]) = 0$, where the unknowns are the entries (b^i_j) of the matrix of φ in the basis $(x_k)_{1 \leqslant k \leqslant 6}$ of the algebra. They are all built from the same core components. The program first solves linearly as much equations as it can; then we are left with a certain number of non linear equations which do require a second phase, which consists in : human inspection to select some equation and some unknown which gets a necessary value from the equation (with consideration to $\det \varphi$), introduction of that value in the program, computation again of all the equations, iteration of the process. In general, this can be done in a number of steps, completing the program to a specific program for that algebra. In some instances, the second phase is rather tricky (e.g. $\mathfrak{g}_{5,1} \times \mathbb{C}^3$). (Note that the program can be easily adapted as `calculisom**.red` to isomorphisms between two Lie algebras and/or other dimensions.)

Step 3) is performed by the various programs `reducparautommodg**.red`. Those are interactively developed with the main common component the procedure deltaprimemodadg(δ, φ) which, given any nilpotent derivation δ and any automorphism φ, returns the nilpotent derivation mod $\operatorname{ad}(\mathfrak{g})$ obtained by first changing δ to $\varphi\delta\varphi^{-1}$ and then killing the "adjoint" parameters as above. To exploit the projectivity of projective equivalence, when (in the separating of cases) some nonzero parameters can be set to 1 by suitable choice of certain b^i_js, we in fact set all those parameters to k ($k \neq 0$) getting one more degree of freedom. Then on some occurrences, we may choose a suitable value for k to kill one more parameter (e.g. $\mathfrak{g}_{6,6}$ N1).

Step 4) is performed by the programs `calculderivgtildedelta**.red`, which have been interactively created from common components as follows. After introduction of the generic nilpotent derivation mod $\operatorname{ad}(\mathfrak{g})$ and the specific values (initial data) for the δ to be considered, a first procedure computes the matrix $matD$ (in the basis $(x_0 = \delta, x_1, ..., x_6)$) of the general element $D \in \operatorname{Der} \tilde{\mathfrak{g}}_\delta$ in pretty much the same way as step 2 (in the presence of continuous parameters in δ, a second phase is necessary). The matrix $matD$ depends on certain of its entries (which are arbitrary complex parameters) $d(i, j)$, $(i, j) \in J = listparametersmatD$, and $\dim \operatorname{Der} \tilde{\mathfrak{g}}_\delta = \# J$. For $(i, j) \in J$, let $D(i, j)$ the matrix with $d(i, j) = 1$, others 0. There is a procedure calcul_t1(i, j) which returns the matrix $t_1 = D(i, j)$ and its commutant $\operatorname{Com}(t_1)$ in $\operatorname{Der} \tilde{\mathfrak{g}}_\delta$. By direct inspection of $matD$, we select $(i, j) \in J$ such that t_1 is a good candidate to be semisimple (very often $(i, j) = (0, 0)$) and check if it actually is, either by hand or the *Reduce* `mateigen` operator. A first change of basis may be necessary to diagonalize t_1. If there is no semisimple element linearly independent of t_1 in $\operatorname{Com}(t_1)$, then $\mathbb{C}t_1$ is a maximal torus and $\operatorname{rk} \tilde{\mathfrak{g}}_\delta = 1$. If not, we look for such a semisimple $t_2 \in \operatorname{Com}(t_1)$ in the same way with the procedure calcul_t2(i, j) which is similar to the first one, yet computes with the parameters of $\operatorname{Com}(t_1)$ and returns $t_2 = D(i, j) \in \operatorname{Com}(t_1)$, along with the joint commutant $\mathrm{Com}(t_1, t_2)$. Then, iterating if necessary with similar procedures in order 3 and 4, we finally get the rank and a maximal torus. A second change of basis may be necessary to diagonalize the whole torus. If no semisimple t_1 is apparent, we check whether $\tilde{\mathfrak{g}}_\delta$ is characteristically nilpotent by computing $(matD)^k$ $(2 \leqslant k \leqslant 7)$. It may happen that $\tilde{\mathfrak{g}}_\delta$ is not characteristically nilpotent, yet the computed t_1, t_2 are not semisimple (e.g. 6_1case2VII4, $t_1 = D(0,0), t_2 = D(2,2)$), but some combination t_1' of them is (e.g. $t_1 + t_2$). Then we select a new semisimple t_2' in $\mathrm{Com}(t_1')$ (e.g. $t_2' = 4D(2,2) + D(0,0) + 2D(1,0)$) and we go on. Now, writing the new matrix for D in the basis that diagonalizes the torus, the weights are read as the diagonal elements. However, it is often not easy to match the weight system to one in Carles' list: if $\mathrm{rk}\,\tilde{\mathfrak{g}}_\delta$ is not maximal (i.e. not equal to $\dim \tilde{\mathfrak{g}}_\delta/[\tilde{\mathfrak{g}}_\delta, \tilde{\mathfrak{g}}_\delta]$), a suitable selecting of the *fundamental weights* is necessary. Now, if there is more than one algebra under that weight system in Carles' list, we compute independently the adjoint cohomology of $\tilde{\mathfrak{g}}_\delta$ and use it to identify the algebra (if it happens to be a member of one of the pairs that have each same cohomology, we have to try either possibility for isomorphism), and finally compute the isomorphism with the matching algebra in the list. When the weight subspaces are 1-dimensional, that is really easy. If there are multiplicities however, that can be very difficult (e.g. 6_1case2VII, 6_7Xa) and may involve lots of hand calculations to supplement the program `calculisom**.red` when it leaves several nonlinear equations to be solved. Finally the running time of each of the programs `calculderivgtildedelta**.red` is around 4s.

Step 5) The program `calculderivgtildedelta**.red` yields an isomorphism $\Psi_\delta : \mathfrak{g}_{7,*.*} \to \tilde{\mathfrak{g}}_\delta$ from some algebra $\mathfrak{g}_{7,*.*}$ to $\tilde{\mathfrak{g}}_\delta$. Now, if $\tilde{\mathfrak{g}}_{\delta_1}$, $\tilde{\mathfrak{g}}_{\delta_2}$ appear to be both isomorphic to the same $\mathfrak{g}_{7,*.*}$, then $\Psi_{\delta_1}(\Psi_{\delta_2})^{-1} : \tilde{\mathfrak{g}}_{\delta_2} \to \tilde{\mathfrak{g}}_{\delta_1}$ is an isomorphism. If it does stabilize $\mathfrak{g}$, then $\delta_2 \sim \delta_1$. If not, to decide whether or not $\delta_2 \sim \delta_1$, we have to go back to the program `reducparautommodg**.red` with the simplifications due to the simple data for δ_2, δ_1. If it appears not to work, we use the program `calculisom**.red` to find an isomorphism between $\tilde{\mathfrak{g}}_{\delta_2}$ and $\tilde{\mathfrak{g}}_{\delta_1}$ that leaves $\mathfrak{g}$ stable, or to prove that such an isomorphism doesn't exist.

2.7. Presentation of results.

The cross table at the end recapitulates the strain of each of the 30 6-dimensional complex NLAs (including direct products). For each $\mathfrak{g}$ of the 4 examples we show here in detail, we first give the generic nilpotent derivation δ mod ad $\mathfrak{g}$, and the elements of $\mathrm{Aut}\,\mathfrak{g}$. Then we present in tables the successive reductions of the parameters in δ by projective equivalence, and the identification of the Lie algebra $\tilde{\mathfrak{g}}_\delta$. There, we refer to the appropriate file of the companion archive [17] (typically, case1N1 in the table for $\mathfrak{g}_{6,1}$ stands for the file `reducparautommodg6_1case1N1.red` in the directory `6_1strain`; in the same way, case1I stands for `calculderivgtildedelta6_1case1I.red` and test1case2N1 stands for `test1case2N1.red`). Unless otherwise stated in the footnotes, when for a given $\mathfrak{g}$ two distinct δs in the last column of the tables give rise to the same algebra $\tilde{\mathfrak{g}}_\delta$, they *are not* projectively equivalent (that includes the δs corresponding to different values of some continuous parameter a or $\varepsilon = 0, 1$). We dismiss all 7-dimensional direct products. Unless otherwise specified, the notation $\sqrt{z}$ will denote any fixed square root of the complex number z. The abbreviation toms in the tables stands for *then one may suppose*. Inside each strain, the first occurrence of an algebra is underlined, for an easier reading of the tables.

3. $\mathfrak{g}_{6,1}$

Commutation relations for $\mathfrak{g}_{6,1}$: $[x_1, x_2] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = x_6$.
Weight system: $\beta_1, \beta_2, \beta_3, -\beta_1 + \beta_2 + \beta_3, \beta_1 + \beta_2, \beta_2 + \beta_3$.
Generic nilpotent derivation mod $ad(\mathfrak{g})$:

$$\delta = \begin{pmatrix} \xi_1^1 & \xi_2^1 & 0 & 0 & 0 & 0 \\ \xi_1^2 & -\xi_1^1 & 0 & 0 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & \xi_1^1 & -\xi_2^1 & 0 & 0 \\ \xi_1^4 & \xi_2^4 & -\xi_1^2 & -\xi_1^1 & 0 & 0 \\ 0 & 0 & \xi_3^5 & \xi_4^5 & 0 & 0 \\ 0 & 0 & \xi_3^6 & \xi_4^6 & \xi_2^4 - \xi_1^3 & 0 \end{pmatrix} \text{ with } A = \begin{pmatrix} \xi_1^1 & \xi_2^1 \\ \xi_1^2 & -\xi_1^1 \end{pmatrix} \text{ nilpotent.}$$

Aut $\mathfrak{g}_{6,1}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_1^1 u & -b_2^1 u & 0 & 0 \\ b_1^4 & b_2^4 & -b_1^2 u & b_2^2 u & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_2^2 b_1^1 - b_1^2 b_2^1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & b_2^3 b_1^2 - b_1^3 b_2^2 - b_1^4 b_2^1 + b_2^4 b_1^1 & (b_2^2 b_1^1 - b_1^2 b_2^1) u \end{pmatrix}$$

with $u \in \mathbb{C}, u \neq 0, b_2^2 b_1^1 - b_1^2 b_2^1 \neq 0$. We then get 2 cases: case 1. $A \neq 0$, case2. $A = 0$. In the case 1 $A \neq 0$ we may suppose $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We denote δ in the case 1 (resp. 2) by $(\xi_1^3, \xi_2^3; \xi_1^4, \xi_2^4; \xi_3^5, \xi_4^5; \xi_3^6, \xi_4^6)$ (resp. $[\xi_1^3, \xi_2^3; \xi_1^4, \xi_2^4; \xi_3^5, \xi_4^5; \xi_3^6, \xi_4^6]$). We get finally for $\tilde{\mathfrak{g}}_\delta$ 27 nonisomorphic algebras:

$\mathfrak{g}_{6,1}$. **Table 1. Case 1:** $A \neq 0$.
Then one may suppose $\xi_1^1 = \xi_1^2 = 0, \xi_2^1 = 1$ **and moreover** $\xi_2^3 = \xi_2^4 = \xi_4^5 = 0$. **Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_1^4 \neq 0$ (case1N1)	$\xi_1^4 = 1$, $\xi_4^6 = 0$	$\xi_3^5 \neq 0$	$\xi_3^5 = 1$, $\xi_3^6 = 0$, $\xi_1^3 = 0, 1$	$(\varepsilon, 0; 1, 0; 1, 0; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^3)$	$\underline{\mathfrak{g}_{7,1.5}}$ $(\varepsilon = 0)$ (case1I) $\underline{\mathfrak{g}_{7,0.1}}$ $(\varepsilon = 1)$ (case1I2)
		$\xi_3^5 = 0$ (case1N2)	$\xi_1^3 = 0, 1$ $\xi_3^6 = 0$	$(\varepsilon, 0; 1, 0; 0, 0; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^3)$	$\underline{\mathfrak{g}_{7,2.6}}$ $(\varepsilon = 0)$ (case1II) $\underline{\mathfrak{g}_{7,1.03}}$ $(\varepsilon = 1)$ (case1II3)
$\xi_1^4 = 0$ (case1N3)	$\xi_4^6 = 0$	$\xi_3^5 \neq 0$	$\xi_3^5 = 1$, $\xi_1^3 = 0, 1$ $\xi_3^6 = 0$	$(\varepsilon, 0; 0, 0; 1, 0; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^3)$	$\underline{\mathfrak{g}_{7,2.35}}$ $(\varepsilon = 0)$ (case1III) $\underline{\mathfrak{g}_{7,1.11}}$ $(\varepsilon = 1)$ (case1III3)
		$\xi_3^5 = 0$ (case1N4)	$\xi_3^6 = 0, 1$ $\xi_1^3 = 0, 1$	$(\varepsilon, 0; 0, 0; 0, 0; \eta, 0)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_1^3, \eta = \xi_3^6)$	$\underline{\mathfrak{g}_{7,3.5}}$ $(\varepsilon = \eta = 0)$ (case1IV) $\underline{\mathfrak{g}_{7,2.26}}$ $(\varepsilon = 0, \eta = 1)$ (case1IV2) $\underline{\mathfrak{g}_{7,2.37}}$ $(\varepsilon = 1, \eta = 0)$ (case1IV3) $\underline{\mathfrak{g}_{7,1.2(ii)}}$ $(\varepsilon = 1, \eta = 1)$ (case1IV4)

$\mathfrak{g}_{6,1}$. **Table 2. Case 2:** $A = 0$. **Subcase 2.1:** $\begin{pmatrix} \xi_3^5 \\ \xi_4^5 \end{pmatrix} \neq 0$ **(case2N1).**

Then one may suppose $\xi_3^5 = 1,\ \xi_4^5 = 0,\ \xi_3^6 = 0,\ \xi_4^6 = 0$. **Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_2^3 \neq 0$	$\xi_2^3 = 1$ $\xi_1^3 = 0$	$\xi_2^4 \neq 0$	$\xi_2^4 = 1$	$[0,1;a,1;1,0;0,0]$ $a \in \mathbb{C}$ $(a = \xi_1^4)$	$\mathfrak{g}_{7,1.3(i_\lambda)}$ $(\lambda = a)$ $(a \neq 0)$ (case2I) $\mathfrak{g}_{7,1.3(v)}$ $(a = 0)$ (case2I2)
		$\xi_2^4 = 0$	$\xi_1^4 = 0,1$	$[0,1;\varepsilon,0;1,0;0,0]$ $\varepsilon = 0,1$ $(\varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,2.29}$ $(\varepsilon = 0)$ (case2II) $\mathfrak{g}_{7,1.3(iv)}$ $(\varepsilon = 1)$ (case2II2)
$\xi_2^3 = 0$, $\xi_1^3 \neq 0$ (case2N2)	$\xi_1^3 = 1$	$\xi_2^4 \neq -1$	$\xi_1^4 = 0$	$[1,0;0,a;1,0;0,0]$ $a \in \mathbb{C}$, $a \neq -1$ $(a = \xi_2^4)$	$\mathfrak{g}_{7,2.1(i_\lambda)}$ $(\lambda = \frac{a}{a-1})$ $(a \neq 0, \pm 1)$ (case2III) $\mathfrak{g}_{7,2.1(iii)}$ $(a = 0)$ (case2III2) $\mathfrak{g}_{7,2.1(v)}$ $(a = 1)$ (case2III3)
		$\xi_2^4 = -1$	$\xi_1^4 = 0,1$	$[1,0;\varepsilon,-1;1,0;0,0]$ $\varepsilon = 0,1$ $(\varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,2.1(i_\lambda)}$ $(\lambda = \frac{1}{2})$ $(\varepsilon = 0)$ (case2IV) $\mathfrak{g}_{7,1.3(ii)}$ $(\varepsilon = 1)$ (case2IV2)
$\xi_2^3 = 0$, $\xi_1^3 = 0$ (case2N3)	$\longrightarrow\longrightarrow$	$\xi_2^4 \neq 0$	$\xi_1^4 = 0$, $\xi_2^4 = 1$	$[0,0;0,1;1,0;0,0]$	$\mathfrak{g}_{7,2.1(iv)}$ (case2V)
		$\xi_2^4 = 0$	$\xi_1^4 = 0,1$	$[0,0;\varepsilon,0;1,0;0,0]$ $\varepsilon = 0,1$ $(\varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,3.19}$ $(\varepsilon = 0)$ (case2VI) $\mathfrak{g}_{7,2.36}$ $(\varepsilon = 1)$ (case2VI2)

$\mathfrak{g}_{6,1}$. **Table 2. (continued) Case 2:** $A = 0$.

Subcase 2.2: $\begin{pmatrix} \xi_3^5 \\ \xi_4^5 \end{pmatrix} = 0$, $\begin{pmatrix} \xi_3^6 \\ \xi_4^6 \end{pmatrix} \neq 0$ **(case2N4). Then one may suppose** $\xi_3^6 = 1,\ \xi_4^6 = 0$. **Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_2^3 \neq 0$	$\xi_2^3 = 1$, $\xi_1^3 = 0$	$\xi_2^4 \neq 0$	$\xi_2^4 = 1$	$[0,1;a,1;0,0;1,0]$ $a \in \mathbb{C}$ $(a = \xi_1^4)$	$\mathfrak{g}_{7,3.1(i_\lambda)}$ * $(a \neq -2, 0, \frac{1}{4})$ (case2VII) $\mathfrak{g}_{7,3.1(iii)}$ $(a = 0)$ (case2VII2) $\mathfrak{g}_{7,2.2}$ $(a = \frac{1}{4}, -2)$ (case2VII3,4)
		$\xi_2^4 = 0$	$\xi_1^4 = 0,1$	$[0,1;\varepsilon,0;0,0;1,0]$ $\varepsilon = 0,1$ $(\varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,3.15}$ $(\varepsilon = 0)$ (case2VIII) $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = 0)$ $(\varepsilon = 1)$ (case2VIII2)
$\xi_2^3 = 0$ (case2N5)	$\longrightarrow\longrightarrow$	$\xi_2^4 \neq -\xi_1^3$	$\xi_1^4 = 0$, $\xi_1^3 = 0,1$	$[\varepsilon,0;0,a;0,0;1,0]$ $a \in \mathbb{C}$, $\varepsilon = 0,1$ $(a \neq -\varepsilon)$ $(\varepsilon = \xi_1^3,\ a = \xi_2^4)$	$\mathfrak{g}_{7,3.1(iii)}$ $(\varepsilon = 0,\ a \neq 0)$ or $(\varepsilon = 1,\ a = 0)$ (case2IX,IX5)† $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = a)$ $(\varepsilon = 1,\ a \neq 0, -1, \frac{1}{2}, 2)$ (case2IX3)† $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = -1)$ $(\varepsilon = 1,\ a = 1/2)$ (case2IX8) $\mathfrak{g}_{7,2.2}$ $(\varepsilon = 1, a = 2)$ (case2IX4)
		$\xi_2^4 = -\xi_1^3$	$\xi_1^4 = 0,1$ $\xi_1^3 = 0,1$	$[\varepsilon,0;\eta,-\varepsilon;0,0;1,0]$ $\varepsilon, \eta = 0,1$ $(\varepsilon = \xi_1^3,\ \eta = \xi_1^4)$	$\mathfrak{g}_{7,4.3}$ $(\varepsilon = \eta = 0)$ (case2X) $\mathfrak{g}_{7,3.14}$ $(\varepsilon = 0,\ \eta = 1)$ (case2X2) $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = -1)$ $(\varepsilon = 1,\ \eta = 0)$ (case2X3) $\mathfrak{g}_{7,2.2}$ $(\varepsilon = \eta = 1)$ (case2X4)‡

* $\lambda = \frac{(3a-2)\sqrt{1-4a}-2a^2+7a-2}{(a-2)\sqrt{1-4a}+5a-2}$. The denominator is $\neq 0$; changing the sign of the square root changes λ to $1-\lambda$. The range of λ is $\mathbb{C} \setminus (\{0,1\} \cup \{-1, \frac{1}{2}, 2\})$.

† proj. equiv. to some cases $\xi_2^3 \neq 0$ of subcase 2.2 (test1case2N5); e.g. for $a = 1$, case2IX3 $\sim$ case2VIII2 (test1case2N52).

‡ proj. equiv. to case2VII3 $(a = \frac{1}{4})$.

$\mathfrak{g}_{6,1}$. **Table 2. (continued) Case 2:** $A = 0$.

Subcase 2.3: $\begin{pmatrix} \xi_3^5 \\ \xi_4^5 \end{pmatrix} = 0$, $\begin{pmatrix} \xi_3^6 \\ \xi_4^6 \end{pmatrix} = 0$ **(case2N6). Then one may suppose** $\begin{pmatrix} \xi_1^3 \\ \xi_2^3 \end{pmatrix} \neq 0$. **Then** [§]

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_2^3 \neq 0$	$\xi_2^3 = 1$ $\xi_1^3 = 0$	$\xi_2^4 \neq 0$	$\xi_2^4 = 1$	$[0, 1; a, 1; 0, 0; 0, 0]$ $a \in \mathbb{C}$ $(a = \xi_1^4)$	$\mathfrak{g}_{7,3.1(i_\lambda)}$ [¶] $(a \neq 0, \frac{1}{4})$ (case2XI, case2XI4) $\mathfrak{g}_{7,3.1(iii)}$ $(a = 0)$ (case2XI2) $\mathfrak{g}_{7,2.2}$ $(a = \frac{1}{4})$ (case2XI3)
		$\xi_2^4 = 0$	$\xi_1^4 = 0, 1$	$[0, 1; \varepsilon, 0; 0, 0; 0, 0]$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,3.14}$ $(\varepsilon = 0)$ (case2XII) $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = 0)$ $(\varepsilon = 1)$ (case2XII2)
$\xi_2^3 = 0$ (case2N7)	$\xi_1^3 = 1$ $\xi_1^4 = 0, 1$	$\longrightarrow\!\!\longrightarrow$	$\longrightarrow\!\!\longrightarrow$	$[1, 0; \varepsilon, a; 0, 0; 0, 0]$ $a \in \mathbb{C}$, $\varepsilon = 0, 1$ $(a = \xi_2^4, \varepsilon = \xi_1^4)$	$\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = \frac{a}{a-1})$ $(\varepsilon = 0, a \neq 0, 1)$ (case2XIII,XIII4,5,6) or $(\varepsilon = 1, a \neq 0, \pm 1)$ (case2XIII7,10) $\mathfrak{g}_{7,3.1(i_\lambda)}$ $(\lambda = 0)$ $(a = 1)$ (case2XIII3,9) $\mathfrak{g}_{7,3.1(iii)}$ $(a = 0)$ (case2XIII2,8) $\mathfrak{g}_{7,2.2}$ $(\varepsilon = 1, a = -1)$ (case2XIII11)

4. $\mathfrak{g}_{6,5}$

Commut. relations for $\mathfrak{g}_{6,5}$: $[x_1, x_2] = x_4$; $[x_1, x_4] = x_5$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_6$. Weight system: $\beta_1, \beta_2, \beta_1 + \beta_2, \beta_1 + \beta_2, 2\beta_1 + \beta_2, \beta_1 + 2\beta_2$. Generic nilpotent derivation mod $ad(\mathfrak{g})$:

$$\delta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2^5 & \xi_3^5 & 0 & 0 & 0 \\ \xi_1^6 & 0 & \xi_3^6 & -\xi_1^3 & 0 & 0 \end{pmatrix}$$

We denote δ by $(\xi_1^3, \xi_2^3; \xi_2^5, \xi_3^5; \xi_1^6, \xi_3^6)$. The automorphisms of $\mathfrak{g}_{6,5}$ fall into 2 kinds. The first kind is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_2^2 b_1^1 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2 b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_2^4 b_1^1 & b_2^2 {b_1^1}^2 & 0 \\ b_1^6 & b_2^6 & b_3^6 & -(b_1^4 + b_1^3) b_2^2 & 0 & {b_2^2}^2 b_1^1 \end{pmatrix} \tag{2}$$

where $b_1^1 b_2^2 \neq 0$. The second kind is comprised of the matrices

$$\Psi = \begin{pmatrix} 0 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & 0 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_1^2 b_2^1 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & -b_1^2 b_2^1 & -b_1^2 b_2^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4 b_2^1 & 0 & -b_1^2 {b_2^1}^2 \\ b_1^6 & b_2^6 & b_3^6 & b_1^2 (b_2^4 + b_2^3) & -{b_1^2}^2 b_2^1 & 0 \end{pmatrix} \tag{3}$$

[§] All cases in this continued table are proj. equiv. to cases from the preceding continued table (testkcase2N6, k=1,2,3).

[¶] $\lambda = \frac{1+\sqrt{1-4a}}{2}$. Changing sign of sq. root changes λ to $1 - \lambda$. Range of $\lambda : \mathbb{C} \setminus \{0, 1, \frac{1}{2}\}$.

where $b_2^1 b_1^2 \neq 0$. We then get 3 cases (which are not proj. equiv.) according to whether ξ_1^3, ξ_2^3 are both nonzero or exactly one of them is 0 or are both 0. Then taking suitable values for the b_j^is, $(1 \leqslant i, j \leqslant 2)$ in (2) or (3), one may suppose in the case 1 that $\xi_1^3 = \xi_2^3 = 1$ and in the case 2 that $\xi_1^3 = 1, \xi_2^3 = 0$. We get finally for $\tilde{\mathfrak{g}}_\delta$ 15 nonisomorphic algebras:

$\mathfrak{g}_{6,5}$. **Table 1.**

if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_1^3\xi_2^3 \neq 0$ (case1)	$\xi_1^3 = 1$, $\xi_2^3 = 1$, $\xi_2^5 = 0$, $\xi_1^6 = 0$	$(1,1;0,a;0,b)$ * $a, b \in \mathbb{C}$ $(a = \xi_3^5,\ b = \xi_3^6)$	$\underline{\mathfrak{g}_{7,1.19}}$ $(b \neq 0, -1;\ a \neq 0;\ (a+b)^2 + 4a \neq 0)$ (case1I) or $(b = 0,\ a = -1)$ (case1I2) $\underline{\mathfrak{g}_{7,2.26}}$ $(a = b = 0)$ (case1I3) $\underline{\mathfrak{g}_{7,2.35}}$ $((a+b)^2 + 4a = 0,\ ab \neq 0)$ (case1I7) $\underline{\mathfrak{g}_{7,2.34}}$ $(a \neq 0, -1,\ b = -1)$ (case1I5) $\underline{\mathfrak{g}_{7,3.4}}$ $(a = 0,\ b = -1)$ (case1I6)

$\mathfrak{g}_{6,5}$. **Table 2.**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
exactly one of $\xi_1^3, \xi_2^3 \neq 0$ (case2)	$\xi_1^3 = 1$, $\xi_2^3 = 0$, $\xi_1^6 = 0$	$\xi_3^5 \neq 0$	$\xi_3^5 = 1$, $\xi_2^5 = 0$	$(1,0;0,1;0,a)$ $a \in \mathbb{C}$ $(a = \xi_3^6)$	$\mathfrak{g}_{7,1.19}$ $(a \neq -1)$ (case2I) $\mathfrak{g}_{7,2.34}$ $(a = -1)$ (case2I2)
		$\xi_3^5 = 0$	$\xi_2^5 = 0, 1$	$(1,0;\varepsilon,0;0,a)$ $a \in \mathbb{C}$, $\varepsilon = 0, 1$ $(a = \xi_3^6$, $\varepsilon = \xi_2^5)$	$\underline{\mathfrak{g}_{7,2.12}}$ $(\varepsilon = 0,\ a \neq 0, -1)$ (case2II) $\underline{\mathfrak{g}_{7,3.22}}$ $(\varepsilon = 0,\ a = 0)$ (case2II2) $\underline{\mathfrak{g}_{7,3.21}}$ $(\varepsilon = 0,\ a = -1)$ (case2II3) $\underline{\mathfrak{g}_{7,1.3(iv)}}$ $(\varepsilon = 1,\ a \neq 0, -1)$ (case2II4) $\underline{\mathfrak{g}_{7,2.1(v)}}$ $(\varepsilon = 1,\ a = 0)$ (case2II5) $\underline{\mathfrak{g}_{7,2.42}}$ $(\varepsilon = 1,\ a = -1)$ (case2II6)
$\xi_1^3 = 0$, $\xi_2^3 = 0$ (case3)		$\xi_3^5\xi_3^6 \neq 0$	$\xi_3^5 = 1$, $\xi_3^6 = 1$, $\xi_2^5 = 0$, $\xi_1^6 = 0$	$(0,0;0,1;0,1)$	$\underline{\mathfrak{g}_{7,2.29}}$ (case3I)
		exactly one of $\xi_3^5, \xi_3^6 \neq 0$	$\xi_3^5 = 0$, $\xi_3^6 = 1$, $\xi_2^5 = 0, 1$ $\xi_1^6 = 0$	$(0,0;\varepsilon,0;0,1)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_2^5)$	$\underline{\mathfrak{g}_{7,3.13}}$ $(\varepsilon = 0)$ (case3II) $\mathfrak{g}_{7,2.29}$ $(\varepsilon = 1)$ (case3II2)
		$\xi_3^5 = 0$, $\xi_3^6 = 0$	$\xi_2^5 = 0, 1$ $\xi_1^6 = 1$	$(0,0;\varepsilon,0;1,0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_2^5)$	$\underline{\mathfrak{g}_{7,2.27}}$ $(\varepsilon = 0)$ (case3III) $\underline{\mathfrak{g}_{7,2.38}}$ $(\varepsilon = 0)$ (case3III2)

* $(1,1;0,a;0,b)$ and $(1,1;0,a';0,b')$ are projectively equivalent under some first kind automorphism if and only if $a = a', b = b'$; they are projectively equivalent under some second kind automorphism if and only if $a' = -(1+b), b' = -(1+a)$ (`testcase1`). $(a+b)^2+4a$ is invariant under that substitution. In the last column discussion, we take advantage of 2d kind proj. equiv. to reduce the cases. The case $b = 0, a \neq 0, -4$ is 2d kind proj. equiv. to $a' = -1, b' \neq -1, 3$ which is included in case1I except for $a = -1$, hence we consider as case1I2 only $a = -1, b = 0$ which is 2d kind proj. equiv. to itself. The case $a = 0, b = 0$ gives in the same way $a' = -1, b' = -1$ which we hence do not list; the case $a = -4, b = 0$ gives $a' = -1, b' = 3$ which is included in case1I7; the case $a \neq 0, -1, b = -1$ gives $a' = 0, b' \neq 0, -1$ which is complementary to case1I; the case $a = 0, b = -1$ is 2d kind proj. equiv. to itself.

5. $\mathfrak{g}_{6,6}$

Commut. relations for $\mathfrak{g}_{6,6}$: $[x_1, x_2] = x_4$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_5$.
Weight system: $\beta_1, \beta_2, \beta_3, \beta_1 + \beta_2, \beta_1 + 2\beta_2, \beta_2 + \beta_3$.
Generic nilpotent derivation mod $ad(\mathfrak{g})$:

$$\delta = \begin{pmatrix} 0 & \xi_2^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^4 & 0 & 0 & 0 \\ \xi_1^5 & 0 & \xi_3^5 & 0 & 0 & \xi_3^4 \\ \xi_1^6 & 0 & \xi_3^6 & -\xi_1^3 & 0 & 0 \end{pmatrix}.$$

Aut $\mathfrak{g}_{6,6}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_2^2 b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4 b_2^2 & {b_2^2}^2 b_1^1 & b_3^4 b_2^2 \\ b_1^6 & b_2^6 & b_3^6 & -b_1^3 b_2^2 & 0 & b_3^3 b_2^2 \end{pmatrix}$$

where $b_1^1 b_2^2 b_3^3 \neq 0$. We denote δ by $(\xi_2^1; \xi_1^3, \xi_2^3; \xi_3^4; \xi_1^5, \xi_3^5; \xi_1^6, \xi_3^6)$. We get for $\tilde{\mathfrak{g}}_\delta$ 35 nonisomorphic algebras:

$\mathfrak{g}_{6,6}$. **Table 1: Suppose first $\xi_2^1 \neq 0$. Then one may suppose $\xi_2^1 = 1, \xi_2^3 = 0$. Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_3^4 \neq 0$ (N1)	$\xi_3^4 = 1$, $\xi_3^5 = 0$, $\xi_1^5 = 0$	$\xi_1^3 \neq 0$	$\xi_1^3 = 1$, $\xi_3^6 = 0$, $\xi_1^6 = 0, 1$	$(1; 1, 0; 1; 0, 0; \varepsilon, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^6)$	$\underline{\mathfrak{g}_{7,0.2}}$ $(\varepsilon = 1)$ (II) $\underline{\mathfrak{g}_{7,1.1(i_\lambda)}}$ $(\varepsilon = 0)$ $(\lambda = 1)$ (II2)
		$\xi_1^3 = 0$	$\xi_1^6 = 0, 1$ $\xi_3^6 = 0, 1$	$(1; 0, 0; 1; 0, 0; \varepsilon, \eta)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_1^6$, $\eta = \xi_3^6)$	$\underline{\mathfrak{g}_{7,2.1(i_\lambda)}}$ $(\lambda = 1)$ $(\varepsilon = \eta = 0)$ (III) $\underline{\mathfrak{g}_{7,2.31}}$ $(\varepsilon = 0, \eta = 1)$ (III2) $\underline{\mathfrak{g}_{7,1.2(i_\lambda)}}$ $(\lambda = 1)$ $(\varepsilon = 1,\ \eta = 0)$ (III3) or $(\varepsilon = 1,\ \eta = 1)$ (III4)
$\xi_3^4 = 0$, $\xi_1^3 \neq 0$ (N2)	$\xi_1^3 = 1$, $\xi_1^5 = 0$, $\xi_1^6 = 0$,	$\longrightarrow\longrightarrow$	$\xi_3^6 = 0$, $\xi_3^5 = 0, 1$	$(1; 1, 0; 0; 0, \varepsilon; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_3^5)$	$\underline{\mathfrak{g}_{7,2.9}}$ $(\varepsilon = 1)$ (V) $\underline{\mathfrak{g}_{7,2.8}}$ $(\varepsilon = 0)$ (V2)
$\xi_3^4 = 0$, $\xi_1^3 = 0$	$\longrightarrow\longrightarrow$	$\xi_3^6 \neq 0$	$\xi_3^6 = 1$, $\xi_3^5 = 0$, $\xi_1^5 = 0, 1$ $\xi_1^6 = 0, 1$	$(1; 0, 0; 0; \varepsilon, 0; \eta, 1)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_1^5, \eta = \xi_1^6)$	$\underline{\mathfrak{g}_{7,3.3}}$ $(\varepsilon = \eta = 0)$ (VI) $\underline{\mathfrak{g}_{7,2.11}}$ $(\varepsilon = 0,\ \eta = 1)$ (VI2) $\underline{\mathfrak{g}_{7,2.22}}$ $(\varepsilon = 1,\ \eta = 0)$ (VI3) $\underline{\mathfrak{g}_{7,1.18}}$ $(\varepsilon = 1,\ \eta = 1)$ (VI4)
		$\xi_3^6 = 0$, $\xi_1^6 \neq 0$	$\xi_1^6 = 1$, $\xi_1^5 = 0$, $\xi_3^5 = 0, 1$	$(1; 0, 0; 0; 0, \varepsilon; 1, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_3^5)$	$\underline{\mathfrak{g}_{7,1.19}}$ $(\varepsilon = 0)$ (VII) $\underline{\mathfrak{g}_{7,1.9}}$ $(\varepsilon = 1)$ (VII2)
		$\xi_3^6 = 0$, $\xi_1^6 = 0$	$\xi_1^5 = 0, 1$ $\xi_3^5 = 0, 1$	$(1; 0, 0; 0; \varepsilon, \eta; 0, 0)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_1^5, \eta = \xi_3^5)$	$\underline{\mathfrak{g}_{7,3.2}}$ $(\varepsilon = \eta = 0)$ (VIII) $\underline{\mathfrak{g}_{7,2.21}}$ $(\varepsilon = 0,\ \eta = 1)$ (VIII2) $\underline{\mathfrak{g}_{7,2.20}}$ $(\varepsilon = 1,\ \eta = 0)$ (VIII3) $\underline{\mathfrak{g}_{7,1.16}}$ $(\varepsilon = 1,\ \eta = 1)$ (VIII4)

$\mathfrak{g}_{6,6}$. **Table 2: Suppose now $\xi_2^1 = 0$. Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta \cong$
$\xi_3^4 \neq 0$ (N3)	$\xi_3^4 = 1$, $\xi_1^5 = 0$, $\xi_3^5 = 0$, $\xi_3^6 = 0$	$\xi_1^3 \neq 0$	$\xi_1^3 = 1$, $\xi_2^3 = 0$, $\xi_1^6 = 0, 1$	$(0; 1, 0; 1; 0, 0; \varepsilon, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^6)$	$\underline{\mathfrak{g}_{7,2.14}}$ $(\varepsilon = 0)$ (IX2) $\underline{\mathfrak{g}_{7,1.01(i)}}$ $(\varepsilon = 1)$ (IX3)
		$\xi_1^3 = 0$	$\xi_2^3 = 0, 1$ $\xi_1^6 = 0, 1$	$(0; 0, \varepsilon; 1; 0, 0; \eta, 0)$ $\varepsilon, \eta = 0, 1$ $(\eta = \xi_1^6, \varepsilon = \xi_2^3)$	$\underline{\mathfrak{g}_{7,3.10}}$ $(\varepsilon = \eta = 0)$ (XI) $\underline{\mathfrak{g}_{7,2.41}}$ $(\varepsilon = 1, \eta = 0)$ (XI3) $\underline{\mathfrak{g}_{7,3.16}}$ $(\varepsilon = 0, \eta = 1)$ (X) $\mathfrak{g}_{7,1.2(i_\lambda)}$ $(\lambda = 1)$ $(\varepsilon = 1, \eta = 1)$ (X3)
$\xi_3^4 = 0$, $\xi_1^3 \neq 0$ (N4)	$\xi_1^3 = 1$, $\xi_2^3 = 0$, $\xi_1^5 = 0$	$\longrightarrow\longrightarrow$	$\xi_1^6 = 0$, $\xi_3^6 = 0$, $\xi_3^5 = 0, 1$	$(0; 1, 0; 0; 0, \varepsilon; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_3^5)$	$\underline{\mathfrak{g}_{7,3.5}}$ $(\varepsilon = 0)$ (XIII) $\underline{\mathfrak{g}_{7,3.4}}$ $(\varepsilon = 1)$ (XII)
$\xi_3^4 = 0$, $\xi_1^3 = 0$	$\longrightarrow\longrightarrow$	$\xi_3^6 \neq 0$ (N5)	$\xi_3^6 = 1$, $\xi_3^5 = 0$, $\xi_1^6 = 0$, $\xi_2^3 = 0, 1$, $\xi_1^5 = 0, 1$	$(0; 0, \varepsilon; 0; \eta, 0; 0, 1)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_2^3, \eta = \xi_1^5)$	direct product $\mathfrak{g}_4 \times \mathfrak{n}$ $(\varepsilon = \eta = 0)$ (XV) $\underline{\mathfrak{g}_{7,3.21}}$ $(\varepsilon = 1, \eta = 0)$ (XIVbis) $\underline{\mathfrak{g}_{7,3.7}}$ $(\varepsilon = 0, \eta = 1)$ (XV3) $\underline{\mathfrak{g}_{7,2.42}}$ $(\varepsilon = 1, \eta = 1)$ (XIVbis2)

$\mathfrak{g}_{6,6}$. **Table 2 continued**

		$\xi_3^6 = 0$, $\xi_1^6 \neq 0$ (N6)	$\xi_1^6 = 1$, $\xi_1^5 = 0$, $\xi_2^3 = 0, 1$ $\xi_3^5 = 0, 1$	$(0; 0, \varepsilon; 0; 0, \eta; 1, 0)$ $\varepsilon, \eta = 0, 1$ $(\varepsilon = \xi_2^3, \eta = \xi_3^5)$	$\underline{\mathfrak{g}_{7,3.11}}$ $(\varepsilon = \eta = 0)$ (XVI) $\underline{\mathfrak{g}_{7,2.36}}$ $(\varepsilon = 0, \eta = 1)$ (XVI2) $\underline{\mathfrak{g}_{7,2.39}}$ $(\varepsilon = 1, \eta = 0)$ (XVI3) $\underline{\mathfrak{g}_{7,1.7}}$ $(\varepsilon = 1, \eta = 1)$ (XVI4)
		$\xi_3^6 = 0$, $\xi_1^6 = 0$, $\xi_3^5 \neq 0$	$\xi_3^5 = 1$, $\xi_1^5 = 0$, $\xi_2^3 = 0, 1$	$(0; 0, \varepsilon; 0; 0, 1; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_2^3)$	$\underline{\mathfrak{g}_{7,3.14}}$ $(\varepsilon = 0)$ (XVII) $\underline{\mathfrak{g}_{7,2.43}}$ $(\varepsilon = 1)$ (XVII2)
		$\xi_3^6 = 0$, $\xi_1^6 = 0$, $\xi_3^5 = 0$, $\xi_2^3 \neq 0$	$\xi_2^3 = 1$, $\xi_1^5 = 0, 1$	$(0; 0, 1; 0; \varepsilon, 0; 0, 0)$ $\varepsilon = 0, 1$ $(\varepsilon = \xi_1^5)$	$\underline{\mathfrak{g}_{7,3.20}}$ $(\varepsilon = 0)$ (XVIII) $\mathfrak{g}_{7,2.39}$ $(\varepsilon = 1)$ (XVIII2)
		$\xi_3^6 = 0$, $\xi_1^6 = 0$, $\xi_3^5 = 0$, $\xi_2^3 = 0$	$\xi_1^5 = 1$	$(0; 0, 0; 0; 1, 0; 0, 0)$	$\underline{\mathfrak{g}_{7,3.8}}$ (XIX)

6. $\mathfrak{g}_{5,1} \times \mathbb{C}$

Commut. relations for $\mathfrak{g}_{5,1} \times \mathbb{C} : [x_1, x_3] = x_5$; $[x_2, x_4] = x_5$. ($\mathfrak{g}_{5,1}$ is the 5-dim. complex Heisenberg Lie algebra.)
Weight system: $\beta_1, \beta_2, \beta_3, \beta_1 - \beta_2 + \beta_3, \beta_1 + \beta_3, \beta_6$.
Generic nilpotent derivation mod $ad(\mathfrak{g})$:

$$\delta = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 & 0 & 0 \\ \xi_1^2 & \xi_2^2 & \xi_4^1 & \xi_4^2 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & -\xi_1^1 & -\xi_1^2 & 0 & 0 \\ \xi_2^3 & \xi_2^4 & -\xi_2^1 & -\xi_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_6^5 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & 0 & 0 \end{pmatrix} \text{ with } A = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 \\ \xi_1^2 & \xi_2^2 & \xi_4^1 & \xi_4^2 \\ \xi_1^3 & \xi_2^3 & -\xi_1^1 & -\xi_1^2 \\ \xi_2^3 & \xi_2^4 & -\xi_2^1 & -\xi_2^2 \end{pmatrix} \text{ nilpotent.}$$

The matrix A has to be a nilpotent element of the Lie algebra $\mathfrak{sp}(4,\mathbb{C})$. Now, from the usual root space decomposition of $\mathfrak{sp}(4,\mathbb{C})$, the set $\mathfrak{sp}(4,\mathbb{C})_+$ of positive elements consists of the matrices $M = \begin{pmatrix} 0 & u^1_2 & u^1_3 & u^1_4 \\ 0 & 0 & u^1_4 & u^2_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -u^1_2 & 0 \end{pmatrix}$, and we know ([11] p.63, 6.2, [2], 14.26) that the nilpotent elements are simply the set $\mathrm{Ad}(G)(\mathfrak{sp}(4,\mathbb{C})_+)$ where G is the Lie group $SP(4,\mathbb{C})$. As $\mathrm{Ad}(P)(X) = PXP^{-1}\ \forall X \in \mathfrak{sp}(4,\mathbb{C})$ and $\mathrm{diag}\,(P, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) \in \mathrm{Aut}(\mathfrak{g}_{5,1}\times\mathbb{C})\ \forall\, P \in SP(4,\mathbb{C})$, we may suppose $A \in \mathfrak{sp}(4,\mathbb{C})_+$, that is $\xi^1_1 = \xi^2_1 = \xi^2_2 = \xi^3_1 = \xi^3_2 = \xi^4_2 = 0$:

$$\delta = \begin{pmatrix} 0 & \xi^1_2 & \xi^1_3 & \xi^1_4 & 0 & 0 \\ 0 & 0 & \xi^1_4 & \xi^2_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\xi^1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi^5_6 \\ \xi^6_1 & \xi^6_2 & \xi^6_3 & \xi^6_4 & 0 & 0 \end{pmatrix}$$

We will denote δ by $(\xi^1_2, \xi^1_3, \xi^1_4; \xi^2_4; \xi^5_6; \xi^6_1, \xi^6_2, \xi^6_3, \xi^6_4)$.

$\mathrm{Aut}(\mathfrak{g}_{5,1}\times\mathbb{C})$ clearly consists of the matrices

$$\begin{pmatrix} & & & & 0 & 0 \\ & & \boxed{s\,\Lambda} & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & b^5_4 & s^2 & b^5_6 \\ b^6_1 & b^6_2 & b^6_3 & b^6_4 & 0 & b^6_6 \end{pmatrix}$$

where $s \in \mathbb{C}^*, \Lambda \in SP(4,\mathbb{C}), b^i_j \in \mathbb{C}, b^6_6 \neq 0$. However, as we need a more explicit expression, we record the following lemma.

Lemma 3. *The automorphisms of* $\mathfrak{g}_{5,1}\times\mathbb{C}$ *fall into 4 kinds.*
(i) First kind.

$$\Phi = \begin{pmatrix} b^1_1 & b^1_2 & b^1_3 & b^1_4 & 0 & 0 \\ * & b^2_2 & * & b^2_4 & 0 & 0 \\ b^3_1 & b^3_2 & b^3_3 & b^3_4 & 0 & 0 \\ * & b^4_2 & * & * & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & b^5_4 & b^3_3 b^1_1 - b^3_1 b^1_3 + b^3_4 b^1_2 - b^3_2 b^1_4 & b^5_6 \\ b^6_1 & b^6_2 & b^6_3 & b^6_4 & 0 & b^6_6 \end{pmatrix}$$

where $b^2_1 = -\frac{(b^2_4 b^1_2 - b^2_2 b^1_4) b^3_1 - b^3_2 b^2_4 b^1_1 + b^3_4 b^2_2 b^1_1}{b^3_3 b^1_1 - b^3_1 b^1_3}$, $b^2_3 = -\frac{(b^2_4 b^1_2 - b^2_2 b^1_4) b^3_3 - b^3_2 b^2_4 b^1_3 + b^3_4 b^2_2 b^1_3}{b^3_3 b^1_1 - b^3_1 b^1_3}$, $b^4_1 = \frac{(b^3_3 b^1_1 - b^3_1 b^1_3)(b^3_2 b^1_1 - b^3_1 b^1_2) - b^4_2 b^3_4 b^2_2 b^1_1 - ((b^2_4 b^1_2 - b^2_2 b^1_4) b^3_1 - b^3_2 b^2_4 b^1_1) b^4_2}{(b^3_3 b^1_1 - b^3_1 b^1_3) b^2_2}$, $b^4_3 = -\frac{(b^3_3 b^1_1 - b^3_1 b^1_3)(b^3_3 b^1_2 - b^3_2 b^1_3) + b^4_2 b^3_4 b^2_2 b^1_3 + ((b^2_4 b^1_2 - b^2_2 b^1_4) b^3_3 - b^3_2 b^2_4 b^1_3) b^4_2}{(b^3_3 b^1_1 - b^3_1 b^1_3) b^2_2}$, $b^4_4 = \frac{b^3_3 b^1_1 - b^3_1 b^1_3 + b^4_2 b^2_4}{b^2_2}$, *and the parameters are subject to the 2 conditions* $\det\Phi \neq 0$ *(that is* $(b^3_4 b^1_2 - b^3_2 b^1_4 + b^3_3 b^1_1 - b^3_1 b^1_3) b^6_6 \neq 0$*) and* $(b^3_3 b^1_1 - b^3_1 b^1_3) b^2_2 \neq 0$.
(ii) Second kind.

$$\Psi = \begin{pmatrix} b^1_1 & b^1_2 & b^1_3 & b^1_4 & 0 & 0 \\ * & 0 & * & b^2_4 & 0 & 0 \\ b^3_1 & b^3_2 & b^3_3 & b^3_4 & 0 & 0 \\ * & -\frac{b^3_3 b^1_1 - b^3_1 b^1_3}{b^2_4} & * & b^4_4 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & b^5_4 & b^3_3 b^1_1 - b^3_1 b^1_3 + b^3_4 b^1_2 - b^3_2 b^1_4 & b^5_6 \\ b^6_1 & b^6_2 & b^6_3 & b^6_4 & 0 & b^6_6 \end{pmatrix}$$

where $b_1^2 = \frac{(b_2^3b_1^1+b_1^3b_2^1)b_4^2}{b_3^3b_1^1-b_1^3b_3^1}$, $b_3^2 = -\frac{(b_3^3b_2^1-b_2^3b_3^1)b_4^2}{b_3^3b_1^1-b_1^3b_3^1}$, $b_1^4 = \frac{(b_3^3b_1^1-b_1^3b_3^1)(b_4^3b_1^1-b_1^3b_4^1)+(b_2^3b_1^1-b_1^3b_2^1)b_4^4b_4^2}{(b_3^3b_1^1-b_1^3b_3^1)b_4^2}$, $b_2^4 = -\frac{b_3^3b_1^1-b_1^3b_3^1}{b_4^2}$, $b_3^4 = \frac{(b_3^3b_1^1-b_1^3b_3^1)(b_4^3b_3^1-b_3^3b_4^1)-(b_3^3b_2^1-b_2^3b_3^1)b_4^4b_4^2}{(b_3^3b_1^1-b_1^3b_3^1)b_4^2}$, *and the parameters are subject to the 2 conditions* $\det\Psi \neq 0$ *(that is* $(b_4^3b_2^1 - b_2^3b_4^1 + b_3^3b_1^1 - b_1^3b_3^1)b_6^6 \neq 0$*) and* $b_3^3b_1^1 - b_1^3b_3^1 \neq 0$.

(iii) Third kind.

$$P = \begin{pmatrix} * & b_2^1 & * & \frac{b_1^4b_3^2-b_3^4b_1^2+b_4^3b_2^1}{b_2^3} & 0 & 0 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & 0 & 0 \\ * & b_2^3 & * & b_4^3 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_4^4 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_4^4b_2^2 - b_2^4b_4^2 + b_3^4b_1^2 - b_1^4b_3^2 & b_6^5 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & b_6^6 \end{pmatrix}$$

where $b_1^1 = -\frac{((b_2^4b_1^2-b_1^4b_2^2)b_3^4+b_4^4b_2^3b_2^1)b_1^2+(b_1^4b_3^2b_2^2+b_4^3b_2^1b_2^2-b_2^3b_4^2b_2^1)b_1^4-(b_1^4b_3^2+b_4^3b_2^1)b_2^4b_1^2}{(b_3^4b_1^2-b_1^4b_3^2)b_2^3}$, $b_3^1 = -\frac{((b_4^3b_2^2-b_2^3b_4^2)b_2^1+b_1^4b_3^2b_2^2)b_3^4+b_4^4b_2^3b_3^2b_2^1-(b_3^4b_2^2-b_2^4b_3^2)b_3^4b_1^2-(b_1^4b_3^2+b_4^3b_2^1)b_2^4b_3^2}{(b_3^4b_1^2-b_1^4b_3^2)b_2^3}$, $b_1^3 = -\frac{(b_4^3b_2^2-b_2^3b_4^2)b_1^4-b_2^4b_4^3b_1^2+b_4^4b_2^3b_1^2}{b_3^4b_1^2-b_1^4b_3^2}$, $b_3^3 = -\frac{(b_4^3b_2^2-b_2^3b_4^2)b_3^4-b_2^4b_4^3b_3^2+b_4^4b_2^3b_3^2}{b_3^4b_1^2-b_1^4b_3^2}$, *and the parameters are subject to the 2 conditions* $\det P \neq 0$ *(that is* $(b_4^4b_2^2 - b_2^4b_4^2 + b_3^4b_1^2 - b_1^4b_3^2)b_6^6 \neq 0$*) and* $(b_3^4b_1^2 - b_1^4b_3^2)b_2^3 \neq 0$.

(iv) Fourth kind.

$$X = \begin{pmatrix} * & b_2^1 & * & b_4^1 & 0 & 0 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & 0 & 0 \\ \frac{b_2^4b_1^2-b_1^4b_2^2}{b_2^1} & 0 & \frac{b_2^4b_3^2-b_3^4b_2^2}{b_2^1} & \frac{b_3^4b_1^2-b_1^4b_3^2}{b_2^1} & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_4^4 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_4^4b_2^2 - b_2^4b_4^2 + b_3^4b_1^2 - b_1^4b_3^2 & b_6^5 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & b_6^6 \end{pmatrix}$$

where $b_1^1 = \frac{(b_4^2b_2^1-b_2^2b_4^1)b_1^4+(b_2^4b_4^1-b_4^4b_2^1)b_1^2}{(b_3^4b_1^2-b_1^4b_3^2)}$, $b_3^1 = \frac{(b_4^2b_2^1-b_2^2b_4^1)b_3^4+(b_2^4b_4^1-b_4^4b_2^1)b_3^2}{(b_3^4b_1^2-b_1^4b_3^2)}$, *and the parameters are subject to the 2 conditions* $\det X \neq 0$ *(that is* $(b_4^4b_2^2 - b_2^4b_4^2 + b_3^4b_1^2 - b_1^4b_3^2)b_6^6 \neq 0$*) and* $(b_3^4b_1^2 - b_1^4b_3^2)b_2^1 \neq 0$.

Proof. Let $\varphi = \left(b_j^i\right)_{1\leqslant i,j\leqslant 6} \in \mathrm{GL}(6,\mathbb{C})$. Denote by

$$ij|k \ (1 \leqslant i < j \leqslant 6,\ 1 \leqslant k \leqslant 6) \tag{4}$$

the equation obtained by projecting on x_k the equation $[\varphi(x_i),\varphi(x_j)] - \varphi([x_i,x_j]) = 0$. We have to solve equations (4). In a first step, we are left with

$$\varphi = \begin{pmatrix} b_1^1 & b_2^1 & b_3^1 & b_4^1 & 0 & 0 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & b_4^3 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_4^4 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & \Delta_{1,3}^{1,3} + \Delta_{1,3}^{2,4} & b_6^5 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & b_6^6 \end{pmatrix}$$

(where $\Delta^{i,j}_{k,\ell}$ is the minor formed with rows i,j and columns k,ℓ in the matrix) and the remaining equations $12|5, 14|5, 23|5, 24|5, 34|5$. The system 1 consisting of the 2 equations $12|5, 14|5$ reads $\Delta^{1,3}_{1,2}+\Delta^{2,4}_{1,2}=0$, $\Delta^{1,3}_{1,4}+\Delta^{2,4}_{1,4}=0$, and has principal determinant $\Delta^{2,4}_{2,4}$ with respect to the principal unknowns b^2_1, b^4_1. The system 2 consisting of the 2 equations $23|5, 34|5$ reads $\Delta^{1,3}_{2,3}+\Delta^{2,4}_{2,3}=0$, $\Delta^{1,3}_{3,4}+\Delta^{2,4}_{3,4}=0$, and has principal determinant $\Delta^{1,3}_{2,4}$ with respect to the principal unknowns b^1_3, b^3_3. The last equation $24|5$ reads $\Delta^{1,3}_{2,4}+\Delta^{2,4}_{2,4}-(\Delta^{1,3}_{1,3}+\Delta^{2,4}_{1,3})=0$. Hence, as $\Delta^{1,3}_{1,3}+\Delta^{2,4}_{1,3}$ has to be nonzero, $\Delta^{1,3}_{2,4}+\Delta^{2,4}_{2,4}$ has to be nonzero as well. Hence at least one of the 2 above systems is a Cramer system.

Case 1. $\Delta^{2,4}_{2,4}\neq 0$. From system 1: $b^2_1 = -\frac{(b^2_4b^1_2-b^2_2b^1_4)b^3_1-b^3_2b^2_4b^1_1+b^3_4b^2_2b^1_1}{\Delta^{2,4}_{2,4}}$, $b^4_1 = -\frac{(b^3_4b^1_1-b^3_1b^1_4)b^4_2-(b^3_2b^1_1+b^3_1b^1_2)b^4_4}{\Delta^{2,4}_{2,4}}$. Now, note that the system 2 has principal determinant $\Delta^{2,4}_{2,4}$ with respect to the principal unknowns b^2_3, b^4_3. Hence we get: $b^2_3 = -\frac{b^3_3b^2_4b^1_2-b^3_3b^2_2b^1_4-b^3_2b^2_4b^1_3+b^3_4b^2_2b^1_3}{\Delta^{2,4}_{2,4}}$, $b^4_3 = -\frac{(b^3_4b^1_3-b^3_3b^1_4)b^4_2+(b^3_3b^1_2-b^3_1b^1_3)b^4_4}{\Delta^{2,4}_{2,4}}$. The last equation $24|5$ now simply reads $\Delta^{1,3}_{1,3}=\Delta^{2,4}_{2,4}$. As b^2_2, b^2_4 cannot simultaneously vanish, one has 2 subcases. Subcase 1.1. $b^2_2\neq 0$. From $24|5$, $b^4_4=\frac{b^3_3b^1_1-b^3_1b^1_3+b^4_2b^2_4}{b^2_2}$. Then we get the first kind. Subcase 1.2. $b^2_2=0, b^2_4\neq 0$. From $24|5$, $b^4_2=-\frac{b^3_3b^1_1-b^3_1b^1_3}{b^2_4}$. Then we get the second kind.

Case 2. $\Delta^{1,3}_{2,4}\neq 0$. From system 2: $b^1_3 = \frac{(b^2_4b^1_2-b^2_2b^1_4)b^4_3+(b^4_2b^1_4-b^4_4b^1_2)b^2_3}{\Delta^{1,3}_{2,4}}$, $b^3_3 = \frac{(b^3_2b^2_4-b^3_4b^2_2)b^4_3-(b^4_4b^3_2-b^4_2b^3_4)b^2_3}{\Delta^{1,3}_{2,4}}$. Now, note that the system 1 has principal determinant $\Delta^{1,3}_{2,4}$ with respect to the principal unknowns b^1_1, b^3_1. Hence we get: $b^1_1 = \frac{(b^2_4b^1_2-b^2_2b^1_4)b^4_1-(b^4_4b^1_2-b^4_2b^1_4)b^2_1}{\Delta^{1,3}_{2,4}}$, $b^1_1 = -\frac{(b^3_4b^2_2-b^3_2b^2_4)b^4_1+(b^4_4b^3_2-b^4_2b^3_4)b^2_1}{\Delta^{1,3}_{2,4}}$. Now $\det\varphi = \frac{(\Delta^{2,3}_{2,4}+\Delta^{1,3}_{2,4})^6(\Delta^{2,4}_{1,3})^2b^6_6}{(\Delta^{1,3}_{2,4})^2}$ and hence the last equation $24|5$ simply amounts to $\Delta^{1,3}_{2,4}=\Delta^{2,4}_{1,3}$. As b^3_2, b^1_2 cannot simultaneously vanish, one has 2 subcases. Subcase 2.1. $b^3_2\neq 0$. From $24|5$, $b^1_4=\frac{b^4_1b^2_3-b^4_3b^2_1+b^3_4b^1_2}{b^3_2}$. Then we get the third kind. Subcase 2.2. $b^3_2=0, b^1_2\neq 0$. From $24|5$, $b^3_4=\frac{b^4_3b^2_1-b^4_1b^2_3+}{b^1_2}$. Then we get the fourth kind.

We get for $\tilde{\mathfrak{g}}_\delta$ 15 nonisomorphic algebras:

$\mathfrak{g}_{5,1}\times\mathbb{C}$. **Table 1.** $\xi_2^1\neq 0$.
Then one may suppose $\xi_2^1=1,\ \xi_3^1=0,\ \xi_4^1=0,\ \xi_2^6=0,\ \xi_3^6=0$. **Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta\cong$
$\xi_4^2\neq 0$ (N1)	$\xi_4^2=1$, $\xi_4^6=0$	$\xi_6^5\neq 0$	$\xi_6^5=1$, $\xi_1^6=0,1$	$(1,0,0;1;1;\varepsilon,0,0,0)$ $\varepsilon=0,1$ $(\varepsilon=\xi_1^6)$	$\underline{\mathfrak{g}_{7,2.10}}$ $(\varepsilon=0)$ (I) $\underline{\mathfrak{g}_{7,1.1(ii)}}$ $(\varepsilon=1)$ (I2)
		$\xi_6^5=0$	$\xi_1^6=1$	$(1,0,0;1;0;1,0,0,0)$	$\underline{\mathfrak{g}_{7,2.4}}$ (II)
$\xi_4^2=0$ (N2)		$\xi_6^5\neq 0$	$\xi_6^5=1$, $\xi_1^6=0,1$, $\xi_4^6=0,1$	$(1,0,0;0;1;\varepsilon,0,0,\eta)$ $\varepsilon,\eta=0,1$ $(\varepsilon=\xi_1^6,\eta=\xi_4^6)$	$\underline{\mathfrak{g}_{7,3.1(iii)}}$ $(\varepsilon=0\ ,\ \eta=0)$ (III) $\underline{\mathfrak{g}_{7,2.1(ii)}}$ $(\varepsilon=1\ ,\ \eta=0)$ $(\varepsilon=0\ ,\ \eta=1)$ (III2,3) [*] $\underline{\mathfrak{g}_{7,1.3(iii)}}$ $(\varepsilon=1\ ,\ \eta=1)$ (III4)
		$\xi_6^5=0$, $\xi_1^6\neq 0$	$\xi_1^6=1$, $\xi_4^6=0,1$	$(1,0,0;0;0;1,0,0,\varepsilon)$ $\varepsilon=0,1$ $(\varepsilon=\xi_4^6)$	$\underline{\mathfrak{g}_{7,3.22}}$ $(\varepsilon=0)$ (IV)[†] $\underline{\mathfrak{g}_{7,2.12}}$ $(\varepsilon=1)$ (IV2)
		$\xi_6^5=0$, $\xi_1^6=0$	$\xi_4^6=1$	$(1,0,0;0;0;0,0,0,1)$	$\mathfrak{g}_{7,3.22}$[†] (V)

$\mathfrak{g}_{5,1}\times\mathbb{C}$. **Table 2.** $\xi_2^1=0,\ \xi_3^1\neq 0$ **(N31).**
Then one may suppose $\xi_3^1=1,\ \xi_4^1=0,\ \xi_3^6=0,\ \xi_4^2=0$ [‡]. **Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta\cong$
$\xi_6^5\neq 0$	$\xi_6^5=1$	$\xi_2^6\neq 0$	$\xi_2^6=1$, $\xi_4^6=0$, $\xi_1^6=0,1$	$(0,1,0;0;1;\varepsilon,1,0,0)$ $\varepsilon=0,1$ $(\varepsilon=\xi_1^6)$	$\underline{\mathfrak{g}_{7,2.28}}$ $(\varepsilon=0)$ (X) [§] $\underline{\mathfrak{g}_{7,2.30}}$ $(\varepsilon=1)$ (X2) [¶]
		$\xi_2^6=0$	$\xi_1^6=0,1$ $\xi_4^6=0,1$	$(0,1,0;0;1;\varepsilon,0,0,\eta)$ $\varepsilon,\eta=0,1$ $(\varepsilon=\xi_1^6,\eta=\xi_4^6)$	$\underline{\mathfrak{g}_{7,3.18}}$ $(\varepsilon=0\ ,\ \eta=0)$ (XI) [‖] $\mathfrak{g}_{7,2.30}$ $(\varepsilon=1\ ,\ \eta=0)$ $(\varepsilon=1\ ,\ \eta=1)$ (XI2,4) [¶] $\mathfrak{g}_{7,2.28}$ $(\varepsilon=0\ ,\ \eta=1)$ (XI3) [§]
$\xi_6^5=0$	$\longrightarrow\longrightarrow$	$\xi_2^6\neq 0$	$\xi_2^6=1$, $\xi_4^6=0$, $\xi_1^6=0,1$	$(0,1,0;0;0;\varepsilon,1,0,0)$ $\varepsilon=0,1$ $(\varepsilon=\xi_1^6)$	$\underline{\mathfrak{g}_{7,3.15}}$ $(\varepsilon=0)$ (XII) [**] $\underline{\mathfrak{g}_{7,3.13}}$ $(\varepsilon=1)$ (XII2) [††]
		$\xi_2^6=0$, $\xi_1^6\neq 0$	$\xi_1^6=1$, $\xi_4^6=0,1$	$(0,1,0;0;0;1,0,0,\varepsilon)$ $\varepsilon=0,1$ $(\varepsilon=\xi_4^6)$	$\mathfrak{g}_{7,3.13}$ (XIII,XIII2) [††]
		$\xi_2^6=0$, $\xi_1^6=0$	$\xi_4^6=1$	$(0,1,0;0;0;0,0,0,1)$	$\mathfrak{g}_{7,3.15}$ (XIV) [**]

$\mathfrak{g}_{5,1}\times\mathbb{C}$. **Table 3.** $\xi_2^1=0,\ \xi_3^1=0$.
Then one may suppose [‡‡]: $\xi_4^2=0$, **(N4)** $\xi_4^1=0$ **(N51). Then**

if	toms	if	toms	δ	$\tilde{\mathfrak{g}}_\delta\cong$
$\xi_2^6\neq 0$ (N6)	$\xi_2^6=1$, $\xi_1^6=0$, $\xi_3^6=0$	$\longrightarrow\longrightarrow$	$\xi_4^6=0$ $\xi_6^5=0,1$	$(0,0,0;0;\varepsilon;0,1,0,0)$ $\varepsilon=0,1$ $(\varepsilon=\xi_6^5)$	$\underline{\mathfrak{g}_{7,4.3}}$ $(\varepsilon=0)$ (XIX) $\mathfrak{g}_{7,3.18}$ $(\varepsilon=1)$ (XIX2) [‖]
$\xi_2^6=0$	$\xi_1^6=0$, $\xi_3^6=0$	$\longrightarrow\longrightarrow$	$\xi_4^6=0$, $\xi_6^5=1$	$(0,0,0;0;1;0,0,0,0)$	$\underline{\mathfrak{g}_{7,4.4}}$ (XXII)

[*] These 2 cases are proj. equiv.
[†] These 2 cases are proj. equiv.
[‡] The case $\xi_4^2\neq 0$ is proj. equiv. to some case $\xi_2^1\neq 0, \xi_4^2=0$.
[§] These 2 cases are proj. equiv.
[¶] These 3 cases are proj. equiv.
[‖] These 2 cases are proj. equiv.
[**] These 2 cases are proj. equiv.
[††] These 3 cases are proj. equiv.
[‡‡] The case $\xi_4^2\neq 0$ is proj. equiv. to some above case $\xi_3^1\neq 0, \xi_4^2=0$ (N4), and the case $\xi_4^1\neq 0$ is proj. equiv. to some above case $\xi_2^1\neq 0, \xi_3^1=\xi_4^1=\xi_4^2=0$ (N51).

Cross Table: cross table for strains of complex 6-dimensional nilpotent Lie algebras

	$\mathfrak{g}_{6,1}$	$\mathfrak{g}_{6,2}$	$\mathfrak{g}_{6,3}$	$\mathfrak{g}_{6,4}$	$\mathfrak{g}_{6,5}$	$\mathfrak{g}_{6,6}$	$\mathfrak{g}_{6,7}$	$\mathfrak{g}_{6,8}$	$\mathfrak{g}_{6,9}$	$\mathfrak{g}_{6,10}$	$\mathfrak{g}_{6,11}$	$\mathfrak{g}_{6,12}$	$\mathfrak{g}_{6,13}$	$\mathfrak{g}_{6,14}$	$\mathfrak{g}_{6,15}$	$\mathfrak{g}_{6,16}$	$\mathfrak{g}_{6,17}$	$\mathfrak{g}_{6,18}$	$\mathfrak{g}_{6,19}$	$\mathfrak{g}_{6,20}$	$\mathfrak{g}_{5,1}\times\mathbb{C}$	$\mathfrak{g}_{5,2}\times\mathbb{C}$	$\mathfrak{g}_{5,3}\times\mathbb{C}$	$\mathfrak{g}_{5,4}\times\mathbb{C}$	$\mathfrak{g}_{5,5}\times\mathbb{C}$	$\mathfrak{g}_{5,6}\times\mathbb{C}$	$\mathfrak{g}_4\times\mathbb{C}^2$	$\mathfrak{n}\times\mathfrak{n}$	$\mathfrak{n}\times\mathbb{C}^3$	$\mathbb{C}^6$
#(strain)	27	15	11	17	15	35	28	19	5	17	17	21	6	10	4	11	7	3	5	3	15	40	23	13	16	16	17	8	25	4
$\mathfrak{g}_{7,0.1}$	•																•													
$\mathfrak{g}_{7,0.2}$						•										•														
$\mathfrak{g}_{7,0.3}$																•						•								
$\mathfrak{g}_{7,0.4(\lambda)}$								•¶		•¶	•¶	•¶																		
$\mathfrak{g}_{7,0.5}$										•	•													•						
$\mathfrak{g}_{7,0.6}$							•			•		•																		
$\mathfrak{g}_{7,0.7}$								•															•			•				
$\mathfrak{g}_{7,0.8}$		•						•				•								•										
$\mathfrak{g}_{7,1.01(i)}$						•							•			•			•											
$\mathfrak{g}_{7,1.01(ii)}$											•					•	•					•								
$\mathfrak{g}_{7,1.02}$			•							•		•																		
$\mathfrak{g}_{7,1.03}$	•							•																		•				
$\mathfrak{g}_{7,1.1(i_\lambda)}$			•*			•†	•‡§									•†	•‡													
$\mathfrak{g}_{7,1.1(ii)}$																	•				•									
$\mathfrak{g}_{7,1.1(iii)}$							•							•																
$\mathfrak{g}_{7,1.1(iv)}$							•			•		•																		
$\mathfrak{g}_{7,1.1(v)}$							•						•				•		•											
$\mathfrak{g}_{7,1.1(vi)}$								•												•			•			•				
$\mathfrak{g}_{7,1.2(i_\lambda)}$ $^{\|}$						•†	•$^{\ddagger\|}$								•$^{\|}$								•$^{\ddagger\|}$				•†			
$\mathfrak{g}_{7,1.2(ii)}$	•						•								•								•							
$\mathfrak{g}_{7,1.2(iii)}$			•				•							•									•							
$\mathfrak{g}_{7,1.2(iv)}$							•				•	•											•							
$\mathfrak{g}_{7,1.3(i_\lambda)}$	•§									•¶	•¶	•¶										•*			•*	•§				
$\mathfrak{g}_{7,1.3(ii)}$	•						•																							
$\mathfrak{g}_{7,1.3(iii)}$										•	•	•									•					•				
$\mathfrak{g}_{7,1.3(iv)}$	•			•	•			•																•						
$\mathfrak{g}_{7,1.3(v)}$	•	•									•	•	•													•				
$\mathfrak{g}_{7,1.4}$																•						•								
$\mathfrak{g}_{7,1.5}$	•													•																
$\mathfrak{g}_{7,1.6}$																•													•	
$\mathfrak{g}_{7,1.7}$						•		•														•								
$\mathfrak{g}_{7,1.8}$		•								•	•	•														•				
$\mathfrak{g}_{7,1.9}$						•								•								•								
$\mathfrak{g}_{7,1.10}$																									•		•			
$\mathfrak{g}_{7,1.11}$	•										•								•											
$\mathfrak{g}_{7,1.12}$																			•			•			•					
$\mathfrak{g}_{7,1.13}$																						•			•					
$\mathfrak{g}_{7,1.14}$																							•	•		•				
$\mathfrak{g}_{7,1.15}$																	•								•				•	
$\mathfrak{g}_{7,1.16}$						•																				•			•	
$\mathfrak{g}_{7,1.17}$										•	•	•												•						
$\mathfrak{g}_{7,1.18}$						•								•								•					•			
$\mathfrak{g}_{7,1.19}$				•	•	•	•	•																				•		
$\mathfrak{g}_{7,1.20}$		•						•				•						•												
$\mathfrak{g}_{7,1.21}$		•										•								•				•						

*: $\lambda = 0$. †: $\lambda = 1$. ‡: $\lambda \neq 1$. §: $\lambda \neq 0$. ¶: **any** $\lambda \in \mathbb{C}$. $^{\|}$: $\lambda \neq -1$.

Cross Table continued

	$\mathfrak{g}_{6,1}$	$\mathfrak{g}_{6,2}$	$\mathfrak{g}_{6,3}$	$\mathfrak{g}_{6,4}$	$\mathfrak{g}_{6,5}$	$\mathfrak{g}_{6,6}$	$\mathfrak{g}_{6,7}$	$\mathfrak{g}_{6,8}$	$\mathfrak{g}_{6,9}$	$\mathfrak{g}_{6,10}$	$\mathfrak{g}_{6,11}$	$\mathfrak{g}_{6,12}$	$\mathfrak{g}_{6,13}$	$\mathfrak{g}_{6,14}$	$\mathfrak{g}_{6,15}$	$\mathfrak{g}_{6,16}$	$\mathfrak{g}_{6,17}$	$\mathfrak{g}_{6,18}$	$\mathfrak{g}_{6,19}$	$\mathfrak{g}_{6,20}$	$\mathfrak{g}_{5,1}\times\mathbb{C}$	$\mathfrak{g}_{5,2}\times\mathbb{C}$	$\mathfrak{g}_{5,3}\times\mathbb{C}$	$\mathfrak{g}_{5,4}\times\mathbb{C}$	$\mathfrak{g}_{5,5}\times\mathbb{C}$	$\mathfrak{g}_{5,6}\times\mathbb{C}$	$\mathfrak{g}_4\times\mathbb{C}^2$	$\mathfrak{n}\times\mathfrak{n}$	$\mathfrak{n}\times\mathbb{C}^3$	$\mathbb{C}^6$
$\mathfrak{g}_{7,2.1(i_\lambda)}$	$\bullet^{\ddagger\S}$					$\bullet^{\dagger}$	$\bullet^{\ddagger}$				$\bullet^{\S}$											$\bullet^{*\ \dagger}$			$\bullet^{*}$	$\bullet^{*}$				
$\mathfrak{g}_{7,2.1(ii)}$							•				•	•									•									
$\mathfrak{g}_{7,2.1(iii)}$	•										•	•	•										•							
$\mathfrak{g}_{7,2.1(iv)}$	•	•					•		•	•																				
$\mathfrak{g}_{7,2.1(v)}$	•		•	•	•		•																							
$\mathfrak{g}_{7,2.2}$	•																						•							
$\mathfrak{g}_{7,2.3}$																•														•
$\mathfrak{g}_{7,2.4}$														•							•									
$\mathfrak{g}_{7,2.5}$			•							•																				
$\mathfrak{g}_{7,2.6}$	•																							•		•				
$\mathfrak{g}_{7,2.7}$																									•				•	
$\mathfrak{g}_{7,2.8}$						•																•								
$\mathfrak{g}_{7,2.9}$						•																					•			
$\mathfrak{g}_{7,2.10}$								•										•			•					•				
$\mathfrak{g}_{7,2.11}$						•								•													•		•	
$\mathfrak{g}_{7,2.12}$				•	•			•													•			•						
$\mathfrak{g}_{7,2.13}$		•										•						•						•						
$\mathfrak{g}_{7,2.14}$						•							•			•			•											
$\mathfrak{g}_{7,2.15}$											•					•	•					•								
$\mathfrak{g}_{7,2.16}$																•									•				•	
$\mathfrak{g}_{7,2.17}$			•				•							•									•							
$\mathfrak{g}_{7,2.18}$														•								•					•			
$\mathfrak{g}_{7,2.19}$														•															•	
$\mathfrak{g}_{7,2.20}$						•																			•	•			•	
$\mathfrak{g}_{7,2.21}$						•																							•	
$\mathfrak{g}_{7,2.22}$						•																•				•				
$\mathfrak{g}_{7,2.23}$		•					•		•	•													•							
$\mathfrak{g}_{7,2.24}$							•															•	•		•	•				
$\mathfrak{g}_{7,2.25}$		•										•	•									•			•					
$\mathfrak{g}_{7,2.26}$	•		•	•	•		•																•							
$\mathfrak{g}_{7,2.27}$					•			•														•							•	
$\mathfrak{g}_{7,2.28}$										•											•		•							
$\mathfrak{g}_{7,2.29}$	•				•			•																•				•		
$\mathfrak{g}_{7,2.30}$		•										•									•					•				
$\mathfrak{g}_{7,2.31}$						•					•	•															•			
$\mathfrak{g}_{7,2.32}$										•		•													•				•	
$\mathfrak{g}_{7,2.33}$		•								•	•	•													•		•			
$\mathfrak{g}_{7,2.34}$				•	•			•																•				•		
$\mathfrak{g}_{7,2.35}$	•			•	•		•	•																						
$\mathfrak{g}_{7,2.36}$	•			•		•	•															•								
$\mathfrak{g}_{7,2.37}$	•						•								•															
$\mathfrak{g}_{7,2.38}$					•			•														•								
$\mathfrak{g}_{7,2.39}$						•																							•	
$\mathfrak{g}_{7,2.40}$																						•		•						
$\mathfrak{g}_{7,2.41}$						•									•							•								
$\mathfrak{g}_{7,2.42}$				•	•	•																•					•			
$\mathfrak{g}_{7,2.43}$						•		•																					•	
$\mathfrak{g}_{7,2.44}$																						•	•				•			
$\mathfrak{g}_{7,2.45}$				•			•															•								

*: $\lambda = 0$. †: $\lambda = 1$. ‡: $\lambda \neq 1$. §: $\lambda \neq 0$.

Cross Table continued

	$\mathfrak{g}_{6,1}$	$\mathfrak{g}_{6,2}$	$\mathfrak{g}_{6,3}$	$\mathfrak{g}_{6,4}$	$\mathfrak{g}_{6,5}$	$\mathfrak{g}_{6,6}$	$\mathfrak{g}_{6,7}$	$\mathfrak{g}_{6,8}$	$\mathfrak{g}_{6,9}$	$\mathfrak{g}_{6,10}$	$\mathfrak{g}_{6,11}$	$\mathfrak{g}_{6,12}$	$\mathfrak{g}_{6,13}$	$\mathfrak{g}_{6,14}$	$\mathfrak{g}_{6,15}$	$\mathfrak{g}_{6,16}$	$\mathfrak{g}_{6,17}$	$\mathfrak{g}_{6,18}$	$\mathfrak{g}_{6,19}$	$\mathfrak{g}_{6,20}$	$\mathfrak{g}_{5,1}\times\mathbb{C}$	$\mathfrak{g}_{5,2}\times\mathbb{C}$	$\mathfrak{g}_{5,3}\times\mathbb{C}$	$\mathfrak{g}_{5,4}\times\mathbb{C}$	$\mathfrak{g}_{5,5}\times\mathbb{C}$	$\mathfrak{g}_{5,6}\times\mathbb{C}$	$\mathfrak{g}_4\times\mathbb{C}^2$	$\mathfrak{n}\times\mathfrak{n}$	$\mathfrak{n}\times\mathbb{C}^3$	$\mathbb{C}^6$
$\mathfrak{g}_{7,3.1(i_\lambda)}$	•¶																					•*	•¶				•*			
$\mathfrak{g}_{7,3.1(iii)}$	•								•												•		•							
$\mathfrak{g}_{7,3.2}$						•																			•					•
$\mathfrak{g}_{7,3.3}$						•																			•				•	
$\mathfrak{g}_{7,3.4}$				•	•	•																					•	•		
$\mathfrak{g}_{7,3.5}$	•					•	•	•														•								
$\mathfrak{g}_{7,3.6}$																						•					•		•	
$\mathfrak{g}_{7,3.7}$				•		•	•															•	•					•		
$\mathfrak{g}_{7,3.8}$						•	•															•	•						•	
$\mathfrak{g}_{7,3.9}$				•																		•	•						•	
$\mathfrak{g}_{7,3.10}$		•				•			•	•												•								
$\mathfrak{g}_{7,3.11}$				•		•																•							•	
$\mathfrak{g}_{7,3.12}$			•																			•								
$\mathfrak{g}_{7,3.13}$					•			•													•			•				•		
$\mathfrak{g}_{7,3.14}$	•					•	•																•				•		•	
$\mathfrak{g}_{7,3.15}$	•			•																	•	•	•				•			
$\mathfrak{g}_{7,3.16}$		•				•			•	•																	•			
$\mathfrak{g}_{7,3.17}$		•									•														•				•	
$\mathfrak{g}_{7,3.18}$		•																			•		•							
$\mathfrak{g}_{7,3.19}$	•																					•						•		
$\mathfrak{g}_{7,3.20}$						•																								•
$\mathfrak{g}_{7,3.21}$				•	•	•																					•		•	
$\mathfrak{g}_{7,3.22}$			•	•	•		•														•									
$\mathfrak{g}_{7,3.23}$																						•		•					•	
$\mathfrak{g}_{7,3.24}$			•																			•							•	
$\mathfrak{g}_{7,4.1}$			•																			•							•	
$\mathfrak{g}_{7,4.2}$																						•								•
$\mathfrak{g}_{7,4.3}$	•																				•	•						•	•	
$\mathfrak{g}_{7,4.4}$																					•									

***:** $\lambda = 0, 1$. **¶: any** $\lambda \in \mathbb{C}$.

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