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LIE ALGEBRAS ASSOCIATED WITH TOPOLOGICAL NILPOTENT GROUPS

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A Lie algebra structure is defined on the set of all continuous one-parameter groups of nilpotent topological groups. Extensions are given to some inductive and projective limits.

I. Introduction

In the theory of real Lie groups it is possible to transform many Lie group problems into algebraic problems by utilizing Lie algebras. The Lie algebra of a real Lie group G may be identified with the set $L(G)$ of all continuous one-parameter groups in G. When G is a topological group this set $L(G)$ is well defined and we may ask under what condition is it possible to define on it a Lie algebra structure.

An answer to that question may be useful in the following problem: given a family $(A_i)_{i\in I}$ of skew-adjoint operators in a Hilbert space, give a meaning to the formal finite linear combination $\sum_i \lambda_i A_i$ ($\lambda_i \in R$) or brackets $[A_i, A_j]$ as skew-adjoint operators. In quantum mechanics these operators (observables) appear as generators of strongly continuous one-parameter groups $(\theta_i)_{i\in I}$ contained in a natural group of covariance (or invariance) of a physical system, the latter being a group G of unitary operators on the Hilbert space of states, endowed with a finer topology than the strong convergence topology. If we know how to define a Lie algebra structure on $L(G)$, we then define the linear combinations and the bracket as the generators of the strongly continuous one-parameter groups $t \to \sum_i \lambda_i\theta_i(t)$ and $t \to \{\theta_i, \theta_j\}(t)$, respectively. They may not coincide with the usual definitions given in operator theory, and if they coincide the group may be defined as the exponentiated form of the Lie algebra of operators.

Before passing to the heart of the problem we mention two examples in which the passage from the topological group to the Lie algebra and vice versa for nilpotent groups is important for physical problems.

a) If one is concerned with one-dimensional quantum mechanics with the potential i/x (because of the existence of charges, masses, ...) an obvious problem is to look at the observable algebra generated by the linear momentum (represented by d/dx) and i/x. In this case, the one-parameter groups corresponding to these two operators generate a group G such that if $(G^n)_{n\geq 1}$ is the central decreasing series, there exists many topologies on G such that $\bigcap_{n\geq 1} \overline{G}_n = \{e\}$ (a similar situation appears when one considers the Dirac operator $\sum_{K=1}^{3} \alpha^K \partial_K + i\beta$ and the Coulomb potential ie/r).

b) In Quantum Field Theory the problem of the free field commutation relations is very old (evidently, we speak about a Bose field). When looking for representations of commutation relations of Quantum Field, one immediately remarks that, first, they form a nilpotent Lie algebra, second, the class of representations of the algebra is very large (even in the finite-dimensional case of quantum mechanics) and it is hopeless to try to classify all of them. For this reason, as well as for the commodity of having unitary operators, H. Weyl has postulated the commutation relations of field theory in their exponentiated form. Passing from that form to the Lie algebra structure (which is the only form dictated by field theory) is in this case a very simple consequence of the general result we get in this article.

In this article we prove in Section III that given a nilpotent topological group G, $L(G)$ has a natural Lie algebra structure.

We then extend this result in Section IV to topological groups which are some type of inductive or projective limit of nilpotent topological groups.

II. Definitions and general properties

Given two elements x, y in a group G, we denote by $\{x, y\} = xyx^{-1}y^{-1}$ the commutator of x and y. Given two subsets M and N in G, we denote by $\{M, N\}$ the smallest group containing the set of all commutators $\{x, y\}$ with $x \in M$, $y \in N$.

The central decreasing series of G is defined by $G^1 = G$, $G^2 = \{G, G\}, \ldots, G^n = \{G, G^{n-1}\}, \ldots$ and G is nilpotent if $G^m = \{e\}$ for some m. The smallest number k such that $G^k = \{e\}$ is called the *length* of G. The derived series of G is defined by $D^0G = G, \ldots, D^nG = \{D^{n-1}G, D^{n-1}G\}, \ldots$ and G is solvable if $D^lG = \{e\}$ for some l. In what follows the term 'topological group' means a Hausdorff topological group, and the term 'Lie group' is used for a finite-dimensional real Lie group.

We denote by $L(G)$ the set of all continuous one-parameter groups of a topological group G. Given p elements $\theta_1, \ldots, \theta_p$ in $L(G)$ and a function α: $\{1, \ldots, p\} \to \{1, \ldots, p\}, \ldots\}$. we define

$$\theta_\alpha(t_1, \ldots, t_p) = \theta_{\alpha_1 \ldots \alpha_p}(t_1, \ldots, t_p) = \{\theta_{\alpha_1}(t_1), \ldots, \{\theta_{\alpha_{p-1}}(t_{p-1}), \theta_{\alpha_p}(t_p)\} \ldots\}.$$

We shall utilize the following formulae

$$\{xy, z\} = \{x, \{y, z\}\}\{y, z\}\{x, z\}, \tag{1}$$

$$\{x, yz\} = \{x, y\}\{x, z\}\{\{z, x\}, y\}, \tag{2}$$

with $x, y, z \in G$ implicitly.

We say that $\theta_1, \ldots, \theta_p \in L(G)$ generate G if the smallest group containing $\bigcup_{1 \le i \le p} \theta_i(R)$ is G.

LEMMA 1. *Let G be a topological group and $\theta: R^p \to G$ a function which is a continuous one-parameter group on every variable when the other variables are fixed. Then there exists a unique $\omega \in L(G)$ such that*

$$\theta(t_1, \ldots, t_p) = \omega(t_1 \ldots t_p) \quad \textit{for every} \quad (t_1, \ldots, t_p) \in R^p.$$

The proof of this lemma is straightforward with

$$\omega(t_1 \ldots t_p) = \theta(1, \ldots, 1, t_1 \ldots t_p).$$

LEMMA 2. *Let A, B be connected Lie groups, G a topological group, $u: A \to G$, $v: B \to G$ continuous homomorphisms, where v is, moreover, bijective.*

Then $v^{-1} \circ u: A \to B$ is a continuous homomorphism.

Consider the map $u \times v: A \times B \to G \times G$, and let C be the inverse image of the diagonal. Thus

$$C = \{(x, y) | u(x) = v(y)\} = \{(x, y) | y = v^{-1} \circ u(x)\}$$

is the graph of $v^{-1} \circ u$. Since $u \times v$ is a continuous homomorphism, C is a closed subgroup and hence a Lie group. It is countable at infinity, therefore the first projection $\mathrm{pr}_1: C \to A$ is an isomorphism, and $v^{-1} \circ u = \mathrm{pr}_2 \circ (\mathrm{pr}_1)^{-1}$ is a continuous homomorphism.

As a consequence, if there exists on G a finer connected topology with respect to which G is a Lie group, it is unique and this finer topology has the same continuous one-parameter groups.

In the following we shall utilize these two facts implicitly.

III. Construction of the Lie algebra

DEFINITION 1. An L-solvable topological group G is a solvable topological group generated by a finite number of continuous one-parameter groups and such that if $\theta_1, \theta_2 \in L(D^pG)$, there exists $\alpha_1, \ldots, \alpha_r \in L(D^{p+1}G)$ and real analytic functions $P_1, \ldots, P_r$ on R^2 such that for any real numbers t_1, t_2 we have

$$\theta_{1,2}(t_1, t_2) = \alpha_1(P_1(t_1, t_2)) \ldots \alpha_r(P_r(t_1, t_2)).$$

If G is an L-solvable topological group, then D^pG is also an L-solvable topological group.

LEMMA 3. *Let G be an L-solvable topological group. There exists on G a finer connected topology for which G is a Lie group.*

Denote by $l(G)$ the smallest number such that $D^{l(G)}G = \{e\}$. We shall prove the lemma by induction on $l(G)$. If $l(G) = 0$, this is obvious. Suppose it is proved for $l(G) \le n-1$. If $l(G) = n$, denote by H the group D^1G with its Lie group structure. Choose a Jordan basis $\omega_1, \ldots, \omega_k$ of the Lie algebra $L(H)$.

Then, if $\theta \in L(H)$, it follows from [1], III, §9, n° 6, Proposition 20, that there exist unique analytic functions $P_\theta^1, \dots, P_\theta^k$ such that

$$\theta(t) = \omega_1(P_\theta^1(t)) \dots \omega_k(P_\theta^k(t))$$

for every $t \in R$.

Choose $\theta_1, \dots, \theta_p$ in $L(G)$ generating G and define the function $f: R^{p+k} \to G$ by

$$f(t_1, \dots, t_p, u_1, \dots, u_k) = \theta_1(t_1) \dots \theta_p(t_p)\omega_1(u_1) \dots \omega_k(u_k).$$

f is surjective. Now, there exist unique analytic functions $P_1, \dots, P_k$ on $R^{2(p+k)}$ such that

$$f(t_1, \dots, t_p, u_1, \dots, u_k)f(t_1', \dots, t_p', u_1', \dots, u_k')$$
$$= \theta_1(t_1+t_1') \dots \theta_p(t_p+t_p')\omega_1(P_1(t_i, t_i', u_j, u_j')) \dots \omega_k(P_k(t_i, t_i', u_j, u_j'))$$

for arbitrary real numbers t_i, t_i', u_j, u_j' $(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant k)$.
We define on R^{p+k} the composition law

$$(t_1, \dots, t_p, u_1, \dots, u_k)(t_1', \dots, t_p', u_1', \dots, u_k')$$
$$= (t_1+t_1', \dots, t_p+t_p', P_1(t_i, t_i', u_j, u_j'), \dots, P_k(t_i, t_i', u_j, u_j')).$$

It is easily seen that it defines a Lie group structure L on R^{p+k}, and f is a continuous homomorphism onto G.

This induces a continuous bijective homomorphism from the connected Lie group $L/\mathrm{Ker} f$ onto G.

THEOREM 1. *Let G be a nilpotent topological group generated by a finite number of continuous one-parameter groups. Then there exists on G a finer connected topology for which G is a Lie group.*

According to Lemma 3 it suffices to prove that G is an L-solvable topological group. Suppose the result is proved for nilpotent topological groups of length $k \leqslant n-1$ (this is obvious if $k = 1$). Suppose the length of G is n. We have $G^n = \{e\}$. Suppose G^{p+1} is generated by a finite number of its continuous one-parameter groups and that if $i \geqslant 1$ and $\theta_1, \dots, \theta_{p+i} \in L(G)$, there exist $\alpha_1, \dots, \alpha_r \in L(G^{p+i})$ and $P_1, \dots, P_r \in R(X_1, \dots, X_{p+i})$ such that

$$\theta_{1 \dots p+i}(t_1, \dots, t_{p+i}) = \alpha_1(P_1(t_1, \dots, t_{p+i})) \dots \alpha_r(P_r(t_1, \dots, t_{p+i}))$$

for arbitrary real numbers $t_1, \dots, t_{p+i}$.

Now

$$\theta_{1 \dots p}(t_1+t_1', t_2, \dots, t_p)$$
$$= \theta_{1.1.2 \dots p}(t_1, t_1', t_2, \dots, t_p)\theta_{1 \dots p}(t_1', t_2, \dots, t_p)\theta_{1 \dots p}(t_1, t_2, \dots, t_p). \quad (3)$$

If $\omega_1, \dots, \omega_q$ is a basis of the Lie algebra $L(G^{p+1})$ such that if $i_k = \dim G^k$, $\omega_1, \dots, \omega_{i_k}$ is a basis of $L(G^k)$ $(p+1 \leqslant k < n)$, then it follows from [1], III, §9, n°5, Proposition 17, that there exist $P_1, \dots, P_q \in R[X_1, \dots, X_{p+1}]$ such that

$$\theta_{1.1.2 \dots p}(t_1, t_2, \dots, t_{p+1}) = \omega_1(P_1(t_1, t_2, \dots, t_{p+1})) \dots \omega_q(P_q(t_1, t_2, \dots, t_{p+1}))$$

for arbitrary real numbers $t_1, t_2, \ldots, t_{p+1}$. We have

$$\begin{aligned}\theta_{1,1,2,\ldots,p}(t_1, t_2, \ldots, t_k+t_k', \ldots, t_{p+1}) \\ = \theta_{1,1,2,\ldots,p}(t_1, t_2, \ldots, t_k, \ldots, t_{p+1})\times \\ \times\theta_{1,1,2,\ldots,p}(t_1, t_2, \ldots, t_k', \ldots, t_{p+1})x(t_1, t_2, \ldots, t_k, t_k', \ldots, t_{p+1}),\end{aligned} \tag{4}$$

where $x(t_1, t_2, \ldots, t_k, t_k', \ldots, t_{p+1}) \in G^{p+2}$.

If $\tilde{G}^{p+1}$ is the universal covering of the Lie group G^{p+1} and $\tilde{\omega}_1, \ldots, \tilde{\omega}_q$ are the coverings of $\omega_1, \ldots, \omega_q$ in $\tilde{G}^{p+1}$, define

$$f(t_1, \ldots, t_{p+1}) = \tilde{\omega}_1\big(P_1(t_1, \ldots, t_{p+1})\big) \ldots \tilde{\omega}_q(P_q(t_1, \ldots, t_{p+1})).$$

From (4), the projection of $f(t_1, \ldots, t_{p+1})$ in $\tilde{G}^{p+1}/\tilde{G}^{p+2}$ is a one-parameter group for every variable. Consequently, there exist real numbers $a_{i_{p+2}+s}$ $(1 \leqslant s \leqslant q-i_{p+2})$ such that

$$P_{i_{p+2}+s}(t_1, \ldots, t_{p+1}) = a_{i_{p+2}+s}t_1 \ldots t_{p+1}.$$

Therefore, there exist unique polynomials $P_1', \ldots, P_q'$ such that

$$\begin{aligned}\theta_{1,\ldots,p}(t_1+t_1', t_2, \ldots, t_p) \\ = \theta_{1,\ldots,p}(t_1', t_2, \ldots, t_p)\theta_{1,\ldots,p}(t_1, t_2, \ldots, t_p)\times \\ \times\omega_1\big(P_1'(t_1, t_1', t_2, \ldots, t_p)\big) \ldots \omega_q\big(P_q'(t_1, t_1', t_2, \ldots, t_p)\big),\end{aligned}$$

where

$$P'_{i_{p+2}+s}(t_1, t_1', t_2, \ldots, t_p) = a_{i_{p+2}+s}t_1t_1't_2 \ldots t_p \quad (1 \leqslant s \leqslant q-i_{p+2}).$$

Thus

$$\omega_{i_{p+2}+1}\left(-a_{i_{p+2}+1}\frac{t_1^2t_2 \ldots t_p}{2}\right) \ldots \omega_q\left(-a_q\frac{t_1^2t_2 \ldots t_p}{2}\right)\theta_{1,\ldots,p}(t_1, t_2, \ldots, t_p)$$

is a one-parameter group with respect to the variable t_1 in G^p/G^{p+2}.

Repeating this procedure, we see that there exist homogeneous polynomials $Q_{i_{p+2}+s}(t_1, \ldots, t_p)$ $(1 \leqslant s \leqslant q-i_{p+2})$ of degree $p+1$ such that

$$\omega_{i_{p+2}+1}\big(Q_{i_{p+2}+1}(t_1, \ldots, t_p)\big) \ldots \omega_q\big(Q_q(t_1, \ldots, t_p)\big)\theta_{1,\ldots,p}(t_1, \ldots, t_p)$$

is a one-parameter group for every variable in G^p/G^{p+2}.

Suppose now that there exist polynomials $Q_{i_h+s}(t_1, \ldots, t_p)$ $(1 \leqslant s \leqslant q-i_h)$ such that

$$\varphi(t_1, \ldots, t_p) = \omega_{i_h+1}\big(Q_{i_h+1}(t_1, \ldots, t_p)\big) \ldots \omega_q\big(Q_q(t_1, \ldots, t_p)\big)\theta_{1,\ldots,p}(t_1, \ldots, t_p)$$

is a one-parameter group for every variable in G^p/G^h.

There exist unique $R_1, \ldots, R_q \in R[X_1, \ldots, X_{p+1}]$ such that

$$\begin{aligned}&\varphi(t_1+t_1', t_2, \ldots, t_p) \\ &= \varphi(t_1, t_2, \ldots, t_p)\varphi(t_1', t_2, \ldots, t_p)\omega_1\big(R_1(t_1, t_1', t_2, \ldots, t_p)\big) \ldots \omega_q\big(R_q(t_1, t_1', t_2, \ldots, t_p)\big).\end{aligned}$$

The function

$$f(t_1, t_1', t_2, \ldots, t_p) = \omega_1\big(R_1(t_1, t_1', t_2, \ldots, t_p)\big) \ldots \omega_q\big(R_q(t_1, t_1', t_2, \ldots, t_p)\big)$$

is analytic from R^{p+1} into the Lie group G^{p+1} and takes values in G^{h}, therefore $R_{i_h+s} = 0$ $(1 \leq s \leq q-i_h)$.

Writing $t_1+(t_1'+t_1'') = (t_1+t_1')+t_1''$, one gets

$$\begin{aligned}
&\varphi(t_1'', \dots, t_p)^{-1}\varphi(t_1', \dots, t_p)^{-1}\varphi(t_1, \dots, t_p)^{-1}\varphi(t_1+t_1'+t_1'', \dots, t_p)\\
&\quad= \omega_{i_{h+1}+1}\big(R_{i_{h+1}+1}(t_1, t_1'+t_1'', \dots, t_p)+R_{i_{h+1}+1}(t_1', t_1'', \dots, t_p)\big)\dots\\
&\quad\dots\,\omega_{i_h}\big(R_{i_h}(t_1, t_1'+t_1'', \dots, t_p)+R_{i_h}(t_1', t_1'', \dots, t_p)\big)\cdot x(t_1, t_1', t_1'', \dots, t_p)\\
&\quad= \omega_{i_{h+1}+1}\big(R_{i_{h+1}+1}(t_1+t_1', t_1'', \dots, t_p)+R_{i_{h+1}+1}(t_1, t_1', \dots, t_p)\big)\dots\\
&\quad\dots\,\omega_{i_h}\big(R_{i_h}(t_1+t_1', t_1'', \dots, t_p)+R_{i_h}(t_1, t_1', \dots, t_p)\big)\cdot y(t_1, t_1', t_1'', \dots, t_p),
\end{aligned}$$

where

$$x(t_1, t_1', t_1'', \dots, t_p) = \omega_1\big(T_1(t_1, t_1', t_1'', \dots, t_p)\big)\dots\omega_{i_{h+1}}\big(T_{i_{h+1}}(t_1, t_1', t_1'', \dots, t_p)\big)$$

with $T_1, \dots, T_{i_{h+1}} \in R[X_1, \dots, X_{p+2}]$ and a similar expression for $y(t_1, t_1', t_1'', \dots, t_p)$.

Therefore, since $\omega_1, \dots, \omega_q$ constitute a basis of the Lie algebra $L(G^{p+1})$, one gets

$$R_k(t_1, t_1'+t_1'', \dots, t_p)+R_k(t_1', t_1'', \dots, t_p) = R_k(t_1+t_1', t_1'', \dots, t_p)+R_k(t_1, t_1', \dots, t_p)$$

and

$$R_k(t_1, 0, t_2, \dots, t_p) = R_k(0, t_1, t_2, \dots, t_p) = 0 \quad (i_{h+1} < k \leq i_h).$$

Then we necessarily have

$$\frac{\partial}{\partial t_1'}R_k(t_1, t_1', \dots, t_k) = \left(\frac{\partial}{\partial t_1''}R_k(t_1+t_1', t_1'', \dots, t_p)\right)_{t_1''=0} - \left(\frac{\partial}{\partial t_1''}R_k(t_1', t_1'', \dots, t_p)\right)_{t_1''=0}.$$

Then

$$R_k(t_1, t_1', \dots, t_p) = \sum_j b_{j,k}(t_2, \dots, t_p)\big((t_1+t_1')^j-t_1^j-t_1'^j\big), \quad \text{where} \quad b_{j,k} \in R[X_1, \dots, X_{p-1}].$$

Therefore,

$$\begin{aligned}
&\psi(t_1, \dots, t_p)\\
&\quad= \omega_{i_{h+1}+1}\Big(-\sum_j b_{j,i_{h+1}+1}(t_2, \dots, t_p)t_1^j\Big)\dots\omega_{i_h}\Big(-\sum_j b_{j,i_h}(t_2, \dots, t_p)t_1^j\Big)\cdot\varphi(t_1, \dots, t_p)
\end{aligned}$$

is a one-parameter group with respect to t_1 in G^p/G^{h+1}.

Like for φ, we have

$$\begin{aligned}
&\psi(t_1, t_2+t_2', \dots, t_p)\\
&\qquad= \psi(t_1, t_2, \dots, t_p)\psi(t_1, t_2', \dots, t_p)\cdot\omega_1\big(S_1(t_1, t_2, t_2', \dots, t_p)\big)\dots\\
&\qquad\qquad\dots\,\omega_{i_h}\big(S_{i_h}(t_1, t_2, t_2', \dots, t_p)\big),
\end{aligned} \tag{5}$$

where S_k are polynomials and $S_{i_{h+1}+s}$ $(1 \leq s \leq i_h-i_{h+1})$ have the same properties in the variables t_2, t_2' as $R_{i_{h+1}+s}$ $(1 \leq s \leq i_h-i_{h+1})$ in t_1, t_1'. We shall prove that the polynomials $S_{i_{h+1}+s}$ are homogeneous of degree one in t_1.

Indeed, we have

$$\varphi(2t_1, t_2+t_2', \ldots, t_p) = \varphi(t_1, t_2, \ldots, t_p)^2\varphi(t_1, t_2', \ldots, t_p)^2\cdot$$
$$\cdot\omega_{i_{h-1}+1}(S_{i_{h-1}+1}(2t_1, t_2, t_2', \ldots, t_p)) \ldots \omega_{i_h}(S_{i_h}(2t_1, t_2, t_2', \ldots, t_p))\cdot x(t_1, t_2, t_2', \ldots, t_p) \quad (6)$$

and

$$\varphi(2t_1, t_2+t_2', \ldots, t_p) = (\varphi(t_1, t_2, \ldots, t_p)\varphi(t_1, t_2', \ldots, t_p))^2\cdot$$
$$\cdot\omega_{i_{h-1}+1}(2S_{i_{h-1}}(t_1, t_2, t_2', \ldots, t_p)) \ldots \omega_{i_h}(2S_{i_h}(t_1, t_2, t_2', \ldots, t_p))\cdot y(t_1, t_2, t_2', \ldots, t_p), \quad (7)$$

where $x(t_1, t_2, t_2', \ldots, t_p)$ and $y(t_1, t_2, t_2', \ldots, t_p)$ are expressions of the form

$$\omega_1(P_1(t_1, t_2, t_2', \ldots, t_p)) \ldots \omega_{i_{h-1}}(P_{i_{h-1}}(t_1, t_2, t_2', \ldots, t_p)),$$

where $P_1, \ldots, P_{i_{h-1}}$ are polynomials.

Since $S_{i_{h-1}+s}(t_1, t_2, t_2', \ldots, t_p)$ is symmetric in t_2, t_2', we have

$$\varphi(t_1, t_2, t_3, \ldots, t_p)\varphi(t_1, t_2', t_3, \ldots, t_p)$$
$$= \varphi(t_1, t_2', t_3, \ldots, t_p)\varphi(t_1, t_2, t_3, \ldots, t_p)\cdot z(t_1, t_2, t_2', \ldots, t_p),$$

where $z(t_1, t_2, t_2', \ldots, t_p)$ is an expression analogous to $x(t_1, t_2, t_2', \ldots, t_p)$.

Then it follows from (6) and (7) and from the fact that $\omega_1, \ldots, \omega_q$ is a basis of $L(G^{p+1})$ that

$$S_k(2t_1, t_2, t_2', \ldots, t_p) = 2S_k(t_1, t_2, t_2', \ldots, t_p) \quad (i_{h+1} < k \leqslant i_h).$$

Therefore S_k $(i_{h+1} < k \leqslant i_h)$ are homogeneous of degree one in t_1.

It is then possible as before, by multiplying by φ on the left to have a one-parameter group in G^p/G^{h+1} with respect to the variable t_2, and since S_k $(i_{h+1} < k \leqslant i_h)$ are homogeneous of degree one in t_1, we have again a one-parameter group in t_1.

Iterating this procedure, it is possible to construct polynomials Q_k $(t_1, \ldots, t_p)$ $(i_{h+1} < k \leqslant i_h)$ such that

$$(t_1, \ldots, t_p) \to \omega_{i_{h-1}+1}(Q_{i_{h-1}+1}(t_1, \ldots, t_p)) \ldots \omega_{i_h}(Q_{i_h}(t_1, \ldots, t_p))\varphi(t_1, \ldots, t_p)$$

is a one-parameter group for every variable in G^p/G^{h+1}. Therefore, there exist polynomials $Q_k(t_1, \ldots, t_p)$ $(1 \leqslant k \leqslant q)$ such that

$$(t_1, \ldots, t_p) \to \omega_1(Q_1(t_1, \ldots, t_p)) \ldots \omega_q(Q_q(t_1, \ldots, t_p))\theta_{1,\ldots,p}(t_1, \ldots, t_p)$$

is a continuous one-parameter group in $G^p/G^n = G^p$. According to Lemma 1, it is of the form $\omega(t_1 \ldots t_p)$, with $\omega \in L(G^p)$. So we can write

$$\theta_{1,\ldots,p}(t_1, \ldots, t_p) = \alpha_1(P_1(t_1, \ldots, t_p)) \ldots \alpha_l(P_l(t_1, \ldots, t_p)), \quad (8)$$

where $\alpha_1, \ldots, \alpha_l \in L(G^p)$ and $P_1, \ldots, P_l$ are polynomials.

Consequently, G^p is generated by a finite number of its continuous one-parameter groups.

Thus G^2 is generated by a finite number of its continuous one-parameter groups and $\theta_{1,2}(t_1, t_2) = \alpha_1'(P_1'(t_1, t_2)) \ldots \alpha_m'(P_m'(t_1, t_2))$, with $\alpha_1', \ldots, \alpha_m' \in L(G^2)$ and $P_1', \ldots, P_m' \in R[X_1, X_2]$.

Therefore, G is an L-solvable topological group. ∎

If a group G satisfies the hypotheses of Theorem 1, what we mean by "the Lie group G" will now be clear.

COROLLARY 1. *Let $A_1, \ldots, A_p$ be skew-adjoint operators defined in a Hilbert space H. Denote by D the intersection of the domains of the products $A_{i_1} \ldots A_{i_k}$ ($i_1, \ldots, i_k \in [1, p]$, $k \in N$), by $A'_1, \ldots, A'_p$ the restrictions of $A_1, \ldots, A_p$ to the domain D, and by $\mathfrak{g}$ the real Lie algebra of operators defined on D generated by $A'_1, \ldots, A'_p$.*

Suppose that the group G generated by the unitary one-parameter groups $t \to e^{tA_i}$ ($i = 1, \ldots, p$) is nilpotent.

Then $\mathfrak{g}$ is nilpotent and every $A' \in \mathfrak{g}$ is an essentially skew-adjoint operator on D.

Indeed, G is a topological nilpotent group with respect to the strong convergence topology and is generated by a finite number of continuous one-parameter groups. Denote by L the group G with its Lie group topology. The injection i: $L \to G$ is a continuous unitary representation of L. Since D contains the set of all differentiable vectors of this representations and every $A' \in \mathfrak{g}$ is skew-symmetric, we get the desired result.

At this level, the following proposition is trivial.

PROPOSITION 1. *Let G be a topological nilpotent group, H and K subgroups of G generated respectively by a finite number of their continuous one-parameter groups, L the group generated by $H \cup K$. Then the Lie groups H and K are analytic subgroups of the Lie group L.*

DEFINITION 2. Let f: $G' \to G$ be a continuous homomorphism between two topological groups, the differential df of f is then the mapping df: $L(G') \to L(G)$ defined by $(df(\theta))(t) = f(\theta(t))$ for every $\theta \in L(G')$, $t \in R$.

THEOREM 2. *Let G be a topological nilpotent group. There exists a unique Lie algebra structure on $L(G)$ satisfying the following property:*

(A) *Given any finite-dimensional real Lie group G' and a continuous homomorphism f: $G' \to G$, df: $L(G') \to L(G)$ is a Lie algebra homomorphism.*

Given $\theta_1, \theta_2 \in L(G)$ one puts on the group H generated by θ_1 and θ_2 the associated connected Lie group structure. The Lie algebra structure of $L(H)$ permits us to define $\theta_1 + \theta_2$, $[\theta_1, \theta_2]$ and $a\theta_1$ for every real number a ($a\theta_1$ is simply the continuous one-parameter group $t \to \theta_1(at)$). It follows from Proposition 1 that it defines a Lie algebra structure on $L(G)$. Property (A) and uniqueness then result from Lemma 2.

COROLLARY 2. *Let f: $G' \to G$ be a continuous homomorphism between two topological nilpotent groups. The mapping df: $L(G') \to L(G)$ is a Lie algebra homomorphism.*

IV. Extension to more general groups

DEFINITION 3. We say that a topological group G is *inductively nilpotent* if there exists an increasing sequence $(G_n)_{n \geq 1}$

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$$

of subgroups of G such that $G = \bigcup_{n=1}^{\infty} G_n$, each G_n being a nilpotent group.

LEMMA 4. *Let $G = \bigcup_{n=1}^{\infty} G_n$ be an inductively nilpotent topological group. Given a connected Lie group H and a continuous homomorphism $f: H \to G$, there exists an integer n for which $f(H) \subset \bar{G}_n$.*

The lemma follows directly from the Baire theorem.

THEOREM 3. *Given an inductively nilpotent topological group $G = \bigcup_{n=1}^{\infty} G_n$, there exists a unique Lie algebra structure on $L(G)$ satisfying property* (A).

By Lemma 4, we have $L(G) = \bigcup_{n=1}^{\infty} L(\bar{G}_n)$. We endow each $L(\bar{G}_n)$ with the Lie algebra structure defined in Theorem 2. If $m \leq n$, $L(\bar{G}_m)$ is a subalgebra of $L(\bar{G}_n)$. Therefore $L(G)$ can be endowed with a unique Lie algebra structure for which each $L(\bar{G}_n)$ is a subalgebra of $L(G)$.

Using again Lemma 4, it can be easily proved that this Lie algebra structure on $L(G)$ is the only one satisfying property (A).

COROLLARY 3. *Given two inductively nilpotent topological groups $G = \bigcup_{n=1}^{\infty} G_n$ and $G' = \bigcup_{n=1}^{\infty} G'_n$ and a continuous homomorphism f from G into G', the mapping df is a homomorphism of the Lie algebra $L(G)$ into the Lie algebra $L(G')$.*

DEFINITION 4. We say that a topological group G is *projectively nilpotent* if $\bigcap_{n=1}^{\infty} \bar{G}^n = \{e\}$.

Denote by π_n the projection of G onto $G_n = G/\bar{G}^n$. Let g_{mn} be the continuous homomorphism of G_n onto G_m (for $m \leq n$) defined by $g_{mn}(x\bar{G}^n) = x\bar{G}^m$. The pair (G_n, g_{mn}) is a projective system of topological groups. The projective limit $\tilde{G} = \lim \operatorname{proj} G_n$ is the closed subgroup of $\prod_{n=1}^{\infty} G_n$ consisting of those sequences $(x_n\bar{G}^n)$ such that $x_m\bar{G}^m = x_n\bar{G}^m$ for $m \leq n$.

We denote by g_n the canonical mapping of $\tilde{G}$ into G_n, which is the restriction to $\tilde{G}$ of the nth projection of $\prod_{n=1}^{\infty} G_n$.

The projective limit of the homomorphism π_n (i.e. the continuous homomorphism $\pi: G \to \tilde{G}$ defined by $\pi(x) = (x\bar{G}^n)$) is one-to-one since $\bigcap_{n=1}^{\infty} \bar{G}^n = \{e\}$, and $\pi(G)$ is dense in $\tilde{G}$. However, is should be noted that π is not necessarily onto. It is sufficient to satisfy the following condition to ensure the surjectivity of π [2]:

(P) *G is complete and each neighborhood of the identity contains one $\bar{G}^n$.*

We now suppose that this condition is satisfied. Then π is a topological isomorphism between G and $\tilde{G}$, and $d\pi$: $L(G) \to L(\tilde{G})$ is bijective.

LEMMA 5. *$(L(G_n), dg_{mn})$ is a projective system of sets and $\varphi = \lim \operatorname{proj} dg_n$ is a bijection* *between* $L(\tilde{G})$ *and* lim proj $L(G_n)$.

The proof of this lemma is straightforward.

THEOREM 4. *Given a projectively nilpotent topological group G satisfying* (P), *there exists on $L(G)$ a unique Lie algebra structure such that*:

(B) *For every $n \geq 1$, $d\pi_n$ is a homomorphism of $L(G)$ into the Lie algebra $L(G_n)$.*

This property ensures also that the Lie algebra $L(G)$ satisfies property (A).

By Corollary 1, $(L(G_n), dg_{nm})$ is a projective system of Lie algebras and consequently lim proj $L(G_n)$ has a Lie algebra structure which we transfer to $L(\tilde{G})$ by the bijection φ and then to $L(G)$ by the bijection $d\pi$. Then the Lie algebra $L(G)$ satisfies property (B) since $d\pi_n = dg_n \circ d\pi$. We shall show that $L(G)$ satisfies property (A). Given a continuous homomorphism f from a connected finite-dimensional real Lie group H into G, if $f_n = \pi_n \circ f$, then (df_n) is a projective system of homomorphisms of the Lie algebra $L(H)$ into the Lie algebra $L(G_n)$. Its projective limit is the homomorphism ψ: $L(H) \to$ lim proj $L(G_n)$ defined by $\psi = \varphi \circ d\pi \circ df$ and hence df is a homomorphism.

The uniqueness follows immediately.

COROLLARY 4. *Given two projectively nilpotent topological groups G, G' satisfying* (P), *and a continuous homomorphism $f: G \to G'$, the mapping $df: L(G) \to L(G')$ is a homomorphism.*

Let $f_n: G_n \to G'_n$ be the continuous homomorphism induced by f. (df_n) is a projective system of homomorphisms of $L(G_n)$ into $L(G'_n)$. Denote by ξ its projective limit. As $df = (d\pi')^{-1} \circ \circ \varphi'^{-1} \circ \xi \circ \varphi \circ d\pi$, df is a homomorphism.

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