ANALYTICITY AND LIE GENERATORS

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ABSTRACT. We consider and answer in the negative the question whether, given a Lie group representation, analyticity of a vector for the representatives, in the differentiated representation, of a set of Lie generators of the Lie algebra implies analyticity for the group representation.

0. INTRODUCTION

Let $\mathfrak{g}$ denote a real finite-dimensional Lie algebra $\{X_1, ..., X_r\}$ a set of Lie generators of $\mathfrak{g}$ (i.e., which generate $\mathfrak{g}$ by linear combinations and commutators), and π a representation of $\mathfrak{g}$ by skew-symmetric operators on a dense invariant domain D in the Hilbert space $\mathcal{H}$. One then knows [2]: (i) $\forall X \in \mathfrak{g}$, the set $D_\omega(\pi(X))$ of analytic vectors of $\pi(X)$ is invariant under the action of $\pi(\mathfrak{g})$; (ii) if all elements of D are analytic for each operator $\pi(X_i)$ $1 \leqslant i \leqslant r$, then there exists a unique unitary representation U in $\mathcal{H}$ of the connected simply-connected Lie group G with Lie algebra $\mathfrak{g}$, such that $\forall X \in \mathfrak{g}$, $\pi(X)$ is contained in the skew-adjoint generator $\mathrm{d}U(X)$ of the unitary one-parameter group $t \to U(\mathrm{e}^{tX})$.

When the hypothesis in (ii) is satisfied, one may then ask the following questions:

(1) Are elements of D necessarily analytic for the group representation U; in other words, does analyticity for each $\pi(X_i)$ $1 \leqslant i \leqslant r$ imply analyticity for $\pi(X)$ $\forall X \in \mathfrak{g}$?
(2) For $X \in \mathfrak{g}$, is $D_\omega(\pi(X))$ necessarily invariant under the action of the group representation U?

We here prove that answers to these questions are negative. More precisely, let U denote the quasi-regular representation of SU(2) in $L^2(S_2)$, $\{J_1, J_2, J_3\}$ the basis of $\mathfrak{su}(2)$ with $[J_i, J_k] = \epsilon_{ikl}J_l$. Then we show in Section 2 that there exists C^∞-vectors for U, which are analytic for $\mathrm{d}U(J_1)$ and $\mathrm{d}U(J_2)$, and however are not analytic for $\mathrm{d}U(J_3)$; there also exists vectors in $L^2(S_2)$ which are analytic for $\mathrm{d}U(J_1)$ and $\mathrm{d}U(J_2)$, but which are not C^∞ for U. In Section 1, we prove that the set of analytic vectors for the regular representation of a compact Lie group is the set of analytic functions on the group.

1. REGULAR REPRESENTATION OF A COMPACT LIE GROUP

Let G be a (real finite-dimensional) Lie group. The space of C^∞-vectors of the regular (left or right)

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representation in $L^2(G)$ is the space $C^\infty(G)$ of C^∞-functions; the space of analytic vectors is contained in the space $C^\omega(G)$ of analytic function on G, but for noncompact G these two spaces may not coincide [5]. We prove that, for compact G, there is coincidence.

LEMMA. *Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, $f \in C^\omega(G)$ and $X \in \mathfrak{g}$ considered as a right-invariant differential operator. Then, there exists constants $C, \lambda > 0$ such that*

$$\forall n \in \mathbb{N}, \qquad \forall x \in G \qquad |X^n f(x)| \leqslant C\lambda^n n!$$

Proof. This lemma results of Nelson's analytic domination theorem [4]; however we give here an elementary proof. Identity $\mathfrak{g}$ with $\mathbb{R}^p$ ($p = \dim G$) by choice of a basis. Let W be an open neighborhood of the identity in G such that $Z \mapsto e^Z$ is a analytic diffeomorphism of an open polydisc P centered at 0 $\mathbb{R}^p$ onto W. Fix $x_0 \in G$. Denote by $H(U, V)$ the Hausdorff series, $Q_1 \subset P$ an open polydisc centered at 0 such that H is analytic on $Q_1 \times Q_1$ and $H(Z, Z') \in P \forall Z, Z' \in Q_1$, and let ψ be defined on $Q_1 \times Q_1$ by $\psi(Z, Z') = f(e^{H(Z, Z')}x_0)$. For $X, Y \in Q_1$ $|t| \leqslant 1$, $f(e^{-tX} e^Y x_0) = \psi(-tX, Y)$. Denote by $(\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_p)$ the coordinates in $\mathbb{R}^p \times \mathbb{R}^p$. Then

$$\frac{d^n}{dt^n}\psi(-tX, Y) = \sum_{n_1 + \cdots + n_p = n} \frac{(-1)^n n!}{n_1! \ldots n_p!} \frac{\partial^n \psi}{\partial \xi_1^{n_1} \ldots \partial \xi_p^{n_p}}(-tX, Y) X_1^{n_1} \ldots X_p^{n_p}$$

where $X_1, \ldots, X_p$ are the components of X. Let $Q \subset Q_1$ be a closed polydisc centered at 0 with radii $\leqslant 1$. As ψ is analytic, there exists constants $A, \delta > 0$ such that for $x, y \in Q$, $|t| \leqslant 1$:

$$\left| \frac{\partial^n \psi}{\partial \xi_1^{n_1} \ldots \partial \xi_p^{n_p}}(-tX, Y) \right| \leqslant A\delta^n n! \quad (n_1 + \ldots + n_p = n).$$

Then

$$\left| \frac{d^n}{dt^n}\psi(-tX, Y) \right| \leqslant A(\delta p)^n n!$$

hence $|X^n f(e^Y x_0)| \leqslant A(\delta p)^n n!$ $\forall n \in \mathbb{N}, \forall X, Y \in Q$. By compacity of G, the lemma is proved.

PROPOSITION. *Let G be a compact Lie group, U its left regular representation in $L^2(G)$. The space of analytic vectors of U is $C^\omega(G)$.*

Proof. Denote by $X \to U_\infty(X)$ the representation of $\mathfrak{g}$ on C^∞-vectors of U, and let $f \in C^\omega(G)$. Then, $\forall X \in \mathfrak{g}$, $U_\infty(X)f = Xf$, X being considered as the right invariant differential operator. From the Lemma, there exists $C, \lambda > 0$ such that

$$\|(U_\infty(X))^n f\|_{L^2(G)} \leqslant C \operatorname{vol}(G)\lambda^n n! \quad \forall n \in \mathbb{N}$$

i.e., f is analytic for $U_\infty(X)$, $\forall X \in \mathfrak{g}$.

2. ANALYTICITY FOR LIE GENERATORS VERSUS ANALYTICITY

2.1. Let $G = \mathrm{SU}(2)$, $\mathfrak{g} = \mathfrak{su}(2)$ and $\{J_1, J_2, J_3\}$ the basis of $\mathfrak{g}$ defined by

$$J_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have $[J_1, J_2] = J_3$, $[J_2, J_3] = J_1$, $[J_3, J_1] = J_2$ and the corresponding one-parameter groups are

$$e^{tJ_1} = \begin{pmatrix} \cos t/2 & i \sin t/2 \\ i \sin t/2 & \cos t/2 \end{pmatrix}, \qquad e^{tJ_2} = \begin{pmatrix} \cos t/2 & -\sin t/2 \\ \sin t/2 & \cos t/2 \end{pmatrix}, \qquad e^{tJ_3} = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}.$$

There is only one conjugacy class in $\mathfrak{su}(2)$ and we shall utilize the following formulae:

$$e^{tJ_2} = e^{-(\pi/2)J_1}\, e^{tJ_3}\, e^{(\pi/2)J_1} \tag{1}$$

$$e^{tJ_1} = e^{(\pi/2)J_2}\, e^{tJ_3}\, e^{-(\pi/2)J_2}, \quad (t \in \mathbb{R}). \tag{2}$$

In the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, introduce as usual

$$H_\pm = iJ_1 \mp J_2, \qquad H_3 = iJ_3.$$

Then

$$[H_3, H_\pm] = \pm H_\pm, \qquad [H_+, H_-] = 2H_3.$$

2.2. Denote by U the quasi-regular representation of G in the Hilbert space $\mathcal{H} = L^2(S_2)$

$$(U(x)f)(\xi) = f(x^{-1} \cdot \xi), \quad x \in G, \xi \in S_2, f \in L^2(S_2)$$

with respect to the usual action of G (via SO(3)) on the sphere S_2. Since S_2 is analytically isomorphic with the homogeneous space G/H, $H = \{e^{tJ_3}; t \in \mathbb{R}\}$, the space of C^∞-vectors for U is of the space of C^∞-functions and, from Section 1, the space of analytic vectors is the space $C^\omega(S_2)$ of analytic functions.

2.3. Still denote by J_k $(1 \leqslant k \leqslant 3)$, $H_\pm$, H_3 the operators $U_\infty(J_k)$, $U_\infty(H_\pm)$, $U_\infty(H_3)$ in $\mathcal{H}$, with domain $D = C^\infty(S_2)$, defined by the differential representation U_∞ of $\mathfrak{g}$, and let $\overline{J_k} = \mathrm{d}U(J_k)$ the closure of J_k (skew-symmetric generator of the unitary one-parameter group $t \mapsto U(e^{tJ_k})$).

It is known that $U \cong \oplus_{l \in \mathbb{N}} D^{(l)}$. ($D^{(l)}$, $l \in \frac{1}{2}\mathbb{N}$, unitary irreducible representation of SU(2).)

Let $\{Y_l^m; l \in \mathbb{N}, -l \leqslant m \leqslant l\}$ the orthonormal basis of $\mathcal{H}$ consisting of the spherical harmonics. Then:

$$H_3 Y_l^m = m Y_l^m, \qquad H_+ Y_l^m = \alpha_{m+1} Y_l^{m+1}, \qquad H Y_l^m = \alpha_m Y_l^{m-1} \tag{3}$$

with $\alpha_n^2 = (l+n)(l-n+1), -l \leqslant n \leqslant l.$

2.4. From (3), $\overline{H}_3 = i\overline{J}_3$ is the operator defined on the domain

$$D_{\overline{H}_3} = \left\{ f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m; \quad \sum_{l=0}^{\infty} \sum_{m=-l}^{l} m^2 |a_l^m|^2 < +\infty \right\}$$

by

$$\overline{H}_3 f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} m a_l^m Y_l^m.$$

Hence the space of analytic vectors of $\overline{J}_3$ is

$$D_\omega(\overline{J}_3) = \left\{ f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m \in \mathcal{H}; \ \exists \alpha > 0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |a_l^m|^2 e^{\alpha |m|} < +\infty \right\}$$

2.5. From (1) and (2), $f \in \mathcal{H}$ is analytic for $\overline{J}_2$ (resp. $\overline{J}_1$) if and only if $U(e^{(\pi/2)J_1})f$ (resp. $U(e^{-(\pi/2)J_2})f$) is analytic for $\overline{J}_3$. From [3, p. 85], for all $l \in \mathbb{N}$, the restriction of $U(e^{(\pi/2)J_1})$ to the subspace $\mathcal{H}^{(l)}$ with basis $\{Y_l^m, -l \leqslant m \leqslant l\}$ has matrix in this basis $(P_{m,n}^l(0))_{-l \leqslant m,n \leqslant l}$, where $P_{m,n}^l$ is the generalized spherical function:

$$P_{m,n}^l(\mu) = \frac{(-1)^{l-m} i^{n-m}}{2^l (l-m)!} \sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}} (1-\mu)^{-(n-m)/2} (1+\mu)^{-(n+m)/2} \frac{d^{l-n}}{d\mu^{l-n}} [(1-\mu)^{l-m} (1+\mu)^{l+m}].$$

We shall denote $p_{m,n}^l = P_{m,n}^l(0)$. Then $f = \Sigma_{l=0}^{\infty} \Sigma_{m=-l}^{l} a_l^m Y_l^m \in \mathcal{H}$ is analytic for $\overline{J}_2$ if and only if there exists $\alpha > 0$ such that

$$\sum_{l=0}^{\infty} \sum_{m=l}^{l} \left| \sum_{n=-l}^{l} p_{m,n}^l a_l^n \right|^2 e^{\alpha |m|} < +\infty. \tag{4}$$

Now, denote by $(q_{m,n}^l)_{-l \leqslant m,n \leqslant l}$ the matrix of the restriction of $U(e^{-(\pi/2)J_2}$ to $\mathcal{H}^{(l)}$.

LEMMA.

$$q_{m,n}^l = (-1)^n i^{m+n} p_{m,n}^l.$$

Proof. From (1),

$$e^{-(\pi/2)J_2} = e^{-(\pi/2)J_1} e^{-(\pi/2)J_3} e^{(\pi/2)J_1}.$$

As $J_3 Y_l^m = -im Y_l^m$, the matrix of the restriction of $U(\mathrm{e}^{-(\pi/2)J_3})$ to $\mathcal{H}^{(l)}$ is the diagonal matrix $(i^m)_{-l \leqslant m \leqslant l}$. Hence, we get

$$g_{m,n}^l = \sum_{k=-l}^{l} p_{m,k}^l i^k p_{k,n}^l$$

since $p_{m,n}^l = p_{n,m}^l \,\forall n, m$. Now $\overline{p_{m,k}^l} = (-1)^{k-m} p_{m,k}^l$, hence

$$q_{m,n}^l = (-1)^m \sum_{k=-l}^{l} (-i)^k p_{m,k}^l p_{k,n}^l$$

$$= (-1)^m \mathrm{e}^{-i(m+n)\pi/2} p_{m,n}^l = (-1)^m (-i)^{m+n} p_{m,n}^l$$

by [3, p. 91 formula (28')] (Q.E.D.)

From the lemma, $f = \Sigma_{l=0}^{\infty} \Sigma_{m=-l}^{l} a_l^m Y_l^m \in \mathcal{H}$ is analytic for $\overline{J_1}$ if and only if there exists $\alpha > 0$ such that

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left| \sum_{n=-l}^{l} (-i)^n p_{m,n}^l a_l^n \right|^2 \mathrm{e}^{\alpha|m|} < +\infty. \tag{5}$$

2.6. The analytic vectors for U are those of $B = (1 - \overline{\Delta})^{1/2} = \oplus_{l=0}^{\infty} \sqrt{1 + l(l+1)}$, where $\overline{\Delta}$ is the closure of the Laplacian $\Delta = J_1^2 + J_2^2 + J_3^2$ which here is the Casimir. Hence $f = \Sigma_{l=0}^{\infty} \Sigma_{m=-l}^{l} a_l^m Y_l^m \in \mathcal{H}$ is analytic for U if and only if there exists $\alpha > 0$ such that $\Sigma_{l=0}^{\infty} (\Sigma_{m=-l}^{l} |a_l^m|^2)\, \mathrm{e}^{\alpha l} < +\infty$.

2.7. Let $f = \Sigma_{l=0}^{\infty} \lambda_l Y_l^l \in \mathcal{H}$ ($\Sigma_{l=0}^{\infty} |\lambda_l|^2 < +\infty$). From (4) and (5), f is analytic for $\overline{J_2}$ if and only if it is analytic for $\overline{J_1}$.

LEMMA.

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} |p_{m,l}^l|^2 \,\mathrm{e}^{\alpha|m|} < +\infty, \quad \forall \alpha, 0 < \alpha < 2 \log 2.$$

Proof. We have $|p_{m,l}^l|^2 = 1/2^{2l} \binom{2l}{l+m}$. For $\alpha > 0$, since

$$\sum_{m=-l}^{l} \binom{2l}{l+m} \mathrm{e}^{\alpha|m|} \leqslant \mathrm{e}^{-\alpha l}(1 + \mathrm{e}^{\alpha})^{2l} + \mathrm{e}^{\alpha l}(1 + \mathrm{e}^{-\alpha})^{2l},$$

there exists $C_\alpha > 0$ such that

$$\sum_{m=-l}^{l} \binom{2l}{l+m} \mathrm{e}^{\alpha|m|} \leqslant C_\alpha \,\mathrm{e}^{\alpha l}, \quad \forall l \in \mathbb{N}$$

therefore $\Sigma_{l=0}^{\infty}\Sigma_{m=-l}^{l}|p_{m,l}^{l}|^2\ e^{\alpha|m|}\leqslant C_\alpha\Sigma_{l=0}^{\infty}\ e^{l(\alpha-2\log 2)}$, and the lemma is proved.

From the lemma, any vector $f=\Sigma_{l=1}^{\infty}\lambda_l Y_l^l(\Sigma_{l=1}^{\infty}|\lambda_l|^2<+\infty)$ is analytic for $\bar{J}_1$ and for $\bar{J}_2$.

Take first $\lambda_l = e^{-\sqrt{l}}$. Then f is a C^∞-vector for the representation U, but f is not an analytic vector, in particular f is not analytic for J_3.

Take now $\lambda_l = 1/l$. Then f is not a C^∞-vector for U, however it is analytic for $\bar{J}_2$ and $\bar{J}_1$.

We also deduce the existence of vectors which are analytic for $\bar{J}_3$ and $\bar{J}_2$, but not for $\bar{J}_1$, by utilizing

$$x=\pm\begin{pmatrix} e^{i\pi/4} & i\,e^{i\pi/4} \\ i\,e^{-i\pi/4} & e^{-i\pi/4}\end{pmatrix}\in G$$

such that $A\ d(x)J_k = J_{\sigma(k)}$ with $\sigma=(1,2,3)$ circular permutation.

In particular, there exists vectors f which are analytic for $\bar{J}_3$ and $\bar{J}_2$ though $U(e^{-(\pi/2)J_2})f$ is not analytic for $\bar{J}_3$.

3. CONCLUSION

Analyticity for (the representatives of) a set of Lie generators $\{X_1, \ldots, X_r\}$ being not sufficient to ensure analyticity for the whole Lie algebra, it is necessary to introduce 'mixed conditions' like

$$\|\pi(X_{i_1})\ldots\pi(X_{i_n})\varphi\|\leqslant C^n n!,\quad C>0,\ \forall n\in\mathbb{N},\ 1\leqslant i_1,\ldots,i_n\leqslant r.$$

That such conditions indeed imply analyticity of φ for the whole Lie algebra is still unknown, though the answer is positive for some 'stratified' nilpotent Lie algebras [1].

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REFERENCES

1. Damlakhi, M. and Helffer, B., *Ann. Sc. Ec. Norm. Sup.* **13**, 397–403 (1980).
2. Flato, M. and Simon, J., *J. Funct. Anal.* **13**, 268–276 (1973).
3. Gel'fand, I.M., Minlos, R.A., and Shapiro, Z.Ya., *Representations of the Rotation and Lorentz Groups and their Applications*, Pergamon Press, 1963.
4. Nelson, E., *Ann. Math.* **70**, 572–615 (1959).
5. Poulsen, N.S., *J. Funct. Anal.* **9**, 87–120 (1972).

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